CS6702 – Graph Theory and Applications

UNIT – 1 GRAPH THEORY

INTRODUCTION Graphs – Introduction – Isomorphism – Sub graphs – Walks, Paths, Circuits – Connectedness – Components – Euler graphs – Hamiltonian paths and circuits – Trees – Properties of trees – Distance and centers in tree – Rooted and binary trees.

PART A

1. Define a Graph.

SOLUTION:

A graph is a ordered pair G=(V,E) where, $V = \{v1, v2, v3 \ldots vn\}$ is the vertex set whose elements are *the* vertices or nodes of the graph, denoted by V(G) or just V. E= ={e1,e2,e3...en} is the edge set whose elements are the edges or connections between vertices of the graph, denoted by E(G) or E.

2. Define Isomorphism of graphs

SOLUTION:

Two graphs G and G' are isomorphic if there is a function $f: V(G) \to V(G')$ from the vertices of G to the vertices of G' such that

(i) f is one to one

(ii) f is onto and

(iii) For each pair of vertices u and v of G

$$[u, v] \in E(G) \Leftrightarrow [f(u), f(v)] \in E(G')$$

Any function f with the above three properties is called an isomorphism from G to G'.

3.Define a complete graph.

SOLUTION:

A simple graph in which every pair of distinct vertices are adjacent is called a complete graph. If G has n vertices then the complete graph will be denoted by k_n.

4. Define a regular graph. Can a complete graph be a regular graph?

SOLUTION:

A graph in which all vertices are of equal degree is called a regular graph.
If the degree of each vertex is r, then the graph is called a regular graph of degree r.
Yes, a complete graph is always a regular.
Since, Every vertex of complete graph k_n of degree n-1.

5.State the handshaking theorem.

SOLUTION:

For any graph G with E edges and V vertices $v_1, v_2, \ldots \ldots v_n$,

$$\sum_{i=1}^{n} d(V_i) = 2E.$$

6.Define bipartite graph.

SOLUTION:

A bipartite graph is an undirected graph whose set vertices can be partitioned into two sets M and N is such a way that each edge joins a vertex in M to a vertex in N and no edge joins either two vertices in M or two vertices in N.

7.Define strongly connected graph.

SOLUTION:

A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

8.Is there a simple graph corresponding to the following degree sequences? (i) (1,1,2,3) (ii) (2,2,4,6)

SOLUTION:

(i) No, There are odd number (3) of odd degree vertices, 1, 1 and 3. Hence there exist no graph corresponding to this degree sequence.

(ii) No, Number of vertices in the graph sequence is four and the maximum degree of a vertex is 6 which is not possible as the maximum degree cannot exist one less than the number of vertices.

9. How many edges are there in a graph with 10 vertices each of degree six?

Solution:

Sum of the degree of the 10 vertices is (6) (10) =60

$$2e = 60 \Rightarrow e = 30.$$

10. Show that the sum of degree of all the vertices in a graph G, is even.

PROOF:

Each edge contribute two degree in a graph.

Also, each edge contributes one degree to each of the vertices on which it is incident.

Hence, if there are N edges in G, then

$$2N = d(v_1) + d(v_2) + \cdots \ldots + d(v_N)$$

Thus, 2N is always even.

11. Determine whether each of these sequences is graphic.

(i) (5, 4, 3, 2, 1)

(ii) (3,2,2,1,0)

(iii) (1,1,1,1,1)

SOLUTION:

(i) No, (5+4+3+2+1=15) sum of degree is odd

(ii) Yes,

(iii) No, (1+1+1+1+1=5) sum of degree is odd.

12. Define Cycle Graph.

SOLUTION:

A Cycle graph of order 'n' is a connected graph whose edges form a cycle of length 'n' and denoted by C_n .

13. What is the degree sequence of K_n , where n is a positive integer? Explain your answer.

SOLUTION:

Each of the n vertices is adjacent to each of the other n-1 vertices, so the degree sequence is n-1, n-1, ……n-1 (n terms)

14. Is there a graph with degree sequence (1,3,3,3,7,6,6)

SOLUTION:

No, since the number of vertices with odd degree is odd, a contradiction to the statement: "the number of vertices of odd degree must be even".

15. Define Adjacency matrix

SOLUTION:

Let G (V, E) be a simple graph with n. Vertices ordered from V_1 *to* $V_{n,}$

Then the adjacency matrix $A = [a_{ij}]_{nXn}$ of G is an n X n symmetric matrix defined by the elements.

$$a_{ij} = \begin{cases} 1 & when\ v_i\ is\ adjacent\ to\ v_j \\ 0\ , & otherwise \end{cases}$$

It is denoted by A (G) or A_G.

16. Define Incidence matrix

SOLUTION:

Let G be a graph with n vertices,

Let $V = \{V_1, V_2, \dots\dots V_n\}$ and $E = \{e_1, e_2, \dots\dots . e_n\}$

Define n x m matrix

$$I_G = [m_{ij}]_{n\ X\ m}$$

Where $m_{ij} = \begin{cases} 1, & when\ \ v_i\ is\ incident\ with\ e_j \\ 0, & otherwise \end{cases}$

17. Define Walk, Path and Trail.

SOLUTION:

Let x, y be the vertices in an undirected graph G= (V, E).

Walk : An x-y walk is a alternating sequence *x=x0,e1,x1,e2.......................en-1,xn-1,en,xn=y* of vertices and edges from G, starting at vertex *x* and ending at vertex *y* and involving *n* edges ei={xi-1,xi}, 1≤ i ≤ n. The length of a walk is its number of edges.

Path: If no vertex of the *x-y* walk occurs more than once, then the walk is called a *x-y* path.

Trail: If no edge in the *x-y* walk is repeated, then the walk is called a *x-y* trail.

Circuit: A closed *x-x* trail is called a circuit.

18. What are the necessary and sufficient conditions to determine whether a given graph has an Euler circuit and Euler trail?

SOLUTION:

a) A given graph G will contain an Euler circuit if and only if all the vertices of G are of even degree.

b) A given graph G will contain an Euler trail if and only if it contains at most two vertices of odd degree.

19. What is a Hamilton circuit?

SOLUTION:

If G=(V,E) is a graph or multigraph with |V|≥3,we say that G has a Hamiltonian circuit if there is a circuit in G that contains every vertex in V i.e. a closed walk that traverses every vertex of G exactly once, except the starting vertex, at which the walk terminates. The figure shows an example. A given graph may have more than one Hamiltonian circuit.

20. Define Binary tree. State its properties.

SOLUTION:

A binary tree is defined as a tree in which there is exactly one vertex of degree two, and each of the remaining vertices is of degree one or three. The two properties of the binary tree are

A)The number of vertices *n* in a binary tree is always odd. This is because there is exactly one vertex of even degree, and the remaining *n-1* vertices are of odd degrees.

b) Let *p* be the number of pendant vertices in a binary tree T. Then *n-p-1* is the number of vertices of degree three. Therefore, the number of edges in T equals *n-1* and hence *p=(n+1)/2.*

PART B

1. **Prove that any undirected graph has an even number of vertices of odd degree.**
PROOF:
Let $G = \{V, E\}$ be any graph with 'n' number of vertices and 'e' number of degrees.

Let $V_1, V_2, \dots \dots \dots . V_K$ be the vertices of odd degree and $V_1', V_2', \dots . V_M'$ be the vertices of even degree.

To prove: k is even

WKT $\sum_{i=1}^{n} d(V_i) = 2|E| = 2e$

$\Rightarrow \sum_{i=1}^{k} d(V_i) + \sum_{j=1}^{m} d(V_i') = 2|E| = 2e$

Each of $d(V_j)$ is even $\Rightarrow \sum_{j=1}^{m} d(V_i')$ *and* $2e$ are even numbers
(Being the sum of even numbers)

$\therefore \sum_{i=1}^{k} d(V_i) + an\ even\ number = an\ even\ number$

$\Rightarrow \sum_{i=1}^{k} d(V_i) = an\ even\ number.$

Since, each term $d(V_i)$ is odd.

Therefore, the number of teams in the LHS sum must be even.

$\Rightarrow$ K is even.
Hence the theorem.

2. **A simple graph with at least two vertices has at least two vertices of same degree.**
Proof:
Let G be a simple graph with $n \geq 2$ vertices.

The graph G has no loop and parallel edges.

Hence the degree of each vertex is $\leq n - 1$.

Suppose that all the vertices of G are of different degrees.

Following degrees $0, 1, 2, 3, \dots \dots, n - 1$ are possible for n vertices of G

Let u be the vertex with degree 0. Then u is an isolated vertex.

Let v be the vertex with degree $n - 1$ then v has $n - 1$ adjacent vertices.

Because v is not an adjacent vertex of itself, therefore every vertex of G other than u is an adjacent vertex of G other than u is an adjacent vertex u.

Hence u cannot be an isolated vertex, this contradiction proves that asimple graph contains two vertices of same degree.

3. **Show that the maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.**

PROOF:

The Handshaking Theorem

For any graph G with E edges and V vertices $v_1, v_2, \dots\dots v_n$,

$$\sum_{i=1}^{n} d(V_i) = 2E$$

Proof:

Let $G = G(V, E)$ be any graph where $V = \{v_1, v_2, \dots v_n\}$ and $E = \{e_1, e_2, \dots\dots e_n\}$.

Since, each edges contributes twice as a degree, the sum of the Degree of all vertices in G is twice as the number of edges in G.

(i.e.,) $\sum_{i=1}^{n} d(V_i) = 2E$

$$\Rightarrow \quad d(v_1) + d(v_2) + \cdots . + d(v_n) = 2e \dots\dots (1)$$

Since we know that the maximum degree of each vertex in the graph G can be $(n-1)$.

$\therefore$ $(1) \Rightarrow (n-1) + (n-2) + \cdots .. \text{to } n \text{ terms } = 2e$

$\Rightarrow n(n-1) = 2e$

$\Rightarrow e = \frac{n(n-1)}{2}$.

Hence the maximum number of edge in any simple graph with'n' vertices is $\frac{n(n-1)}{2}$.

4. **If a graph G (either connected or not) has exactly two vertices of odd degree, there is path joining these two vertices,**

PROOF:

Case (i):

Let G be connected.

Let $v_1 \text{ and } v_2$ be the only vertices of G with are of odd degree.

But we know that number of odd vertices is even.

Clearly there is a path connecting $v_1 \text{ and } v_2$, because G isconnected.

Case (ii):

Let G be disconnected.

Then the components of G are connected.

Hence $v_1 \text{ and } v_2$ should belong to the same component of G.

Hence there is a path between $v_1 \text{ and } v_2$.

5. **The maximum number of edges in a simple disconnected graph G with n vertices and k components is $\frac{(n-k)(n-k+1)}{2}$.**

PROOF:

Let the number of vertices in the i^{th} component of G be n_1 $(n_i \geq 1)$

Then $n_1 + n_2 + \cdots \ldots . + n_k = n \quad or \quad \sum_{i=1}^{k} n_i = n$

Hence $\sum_{i=1}^{k}(n_i - 1) = n - k$

$$\therefore \left\{\sum_{i=1}^{k}(n_i - 1)\right\}^2 = n^2 - 2nk + k^2$$

$\Rightarrow \sum_{i=1}^{k}(n_i - 1)^2 + 2\sum_{i \neq j}(n_i - 1)(n_j - 1) = n^2 - 2nk + k^2 \ldots\ldots$ (1)

$\Rightarrow \sum_{i=1}^{k}(n_i - 1)^2 \leq n^2 - 2nk + k^2$ [Since, (1)≥0, as each $n_i \geq 1$]

$\Rightarrow \sum_{i=1}^{k}({n_i}^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$

$\Rightarrow \sum_{i=1}^{k} {n_i}^2 \leq n^2 - 2nk + k^2 + 2n - k$ …………………………..(2)

Now the maximum number of edges in the i^{th} component of

$$G = \frac{1}{2}n_i(n_i - 1)$$

Therefore maximum number of edges of G

$$\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{k} n_i(n_i - 1) \\
&= \frac{1}{2}\sum_{i=1}^{k} {n_i}^2 - \frac{1}{2}n, \quad by\ (1) \\
&\leq \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k) - \frac{1}{2}n, \quad by\ (2) \\
&\leq \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k) \\
&\leq \frac{1}{2}[(n-k)^2 + (n-k)] \\
&\leq \frac{1}{2}[(n-k)(n-k+1)]
\end{aligned}$$

6. **Show that a simple graph G with 'n' vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.**

SOLUTION:

Suppose that G is not connected.
Then it has a component of k vertices for some k, $1 \leq k \leq n-1$.
The most edges G could have is

$$C(k,2) + C(n-k,2) = \frac{[k(k-1) + (n-k)(n-k-1)]}{2}$$
$$= \frac{k^2 - nk + (n^2 - n)}{2}$$

This Quadratic function of f is minimized at $k = \frac{n}{2}$ and maximized at k=1 or k = n-1. Hence, if G is not connected, then the number of edges does not exceed the value of this function at 1 and at n-1, namely $\frac{(n-1)(n-2)}{2}$.

7. **A given connected graph G is an Euler graph if and only if all vertices of G is of even degree.**

PROOF:

Suppose G is an Euler graph.

⇒ G contains an Euler line

⇒ G contains a closed walk covering all edges.

To Prove:

All vertices of G are of even degree.

In training the closed walk, every time the walk meets a vertex v, it goes through two new edges incident on V with one we 'entered' and other 'exited'. This is true for all vertices, because it is a closed walk. Thus the degree of every vertex is even.

Conversely, suppose that all vertices of G are of even degree.

To Prove:

G is an Euler graph.

(i.e.) To Prove: G contains an Euler line.

Construct a closed walk starting at an arbitrary vertex v and going through the edge of G such that no edge is repeated. Because, each vertex is of even degree, we can exit from each end, every vertex where we enter, the tracing can stop only at the vertex v. Name the closed walk as h.

Case (i):

If h covers all edges of G, then h becomes an Euler line, and hence, G is an Euler graph.

Case (ii):

If h does not cover all edges of G then remove all edges of h from g and obtain the remaining graph G', Because both G and G' have all their vertex of even degree.

⇒ Every vertex in G' is also even degree.

Since G is connected, h will touch G' atleast one vertex v', starting from v', we can agin construct a new walk h' inG'. This will terminate only atv', because, every vertex inG' is also of even degree.

Now this walk h'n combined with h forms a closed walk starts and ends at v and has more edges than h. This process is repeated until we obtain a closed walk covering all edges of G. Thus G is an Euler graph.

Hence the proof.

8. **A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.**

SOLUTION:

First Prove the Problem 7.

Given: A connected multigraph G has an Euler path but not an Euler circuit.

To Prove:

G has exactly two vertices of odd degree suppose that a connected multigraph does have an Euler path from a to b, but not an Euler circuit. The first edge of the path contributes one to the degree of a.

A contribution of two to the degree of a is made every time the a path passes through a . The last edge in the path contributes one to the degree of b.

Every time the path goes through b there is a contribution of two to its degree, consequently, both a and b have odd degree. Every other vertex has even degree, because the path contributes two to the degree of a vertex whenever it passes through it.

9. In a complete graph with n vertices there are $\frac{(n-1)}{2}$ edge-disjoint Hamiltoinan circuits, if n is an odd numbers ≥ 3.

Solution:

A complete graph G of n vertices has $\frac{(n-1)}{2}$ edges, and a Hamiltonian circuit in G consists of n edges.

Therefore, the number of edge-disjoint Hamiltonian circuit in G cannot exceed $\frac{(n-1)}{2}$.

That there $\frac{(n-1)}{2}$ **edge-disjoint Hamiltonian circuit**, when n is odd.

The sub graph (of a complete graph of n vertices) in figure is a Hamiltonian circuit keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by $\frac{360}{(n-1)}, 2\frac{360}{(n-1)}, 3\frac{360}{(n-1)}, \ldots \ldots$

Observe that each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones.

Thus we have $\frac{(n-3)}{2}$ new Hamiltonain circuits, all edge disjoint form the one is figure and also edge disjoint among them.

Hence the theorem.

10. Define a tree. Illustrate with example. State and prove any two properties of trees

SOLUTION:

Tree

A tree is a connected graph without any circuits.

Example:

(a) ° (b) ∘ — — — — ∘ (c) ∘ --------∘ -------- ∘ (d) ∘ — — — —∘ — — — —∘ — — — —∘

Trees with one, two, three vertices.

Property: 1

Statement:

There is one and only one path between every pair of vertices in a tree, T

Proof:

Given: T is a tree.

⇒ T is a connected graph without any circuits.

To Prove: There is one and only one path between every pair of vertices in T.

Proof:

Now assume that there are two distinct paths between a pair of vertices a and b in T.

⇒ $a \cup b$ Contain a circuit which is a contradiction.

⇒ T cannot be a tree.

So, our assumption is wrong.

∴ There is one and only one path between every pair of vertices in a tree, T

Property: 2

Statement:
Every non-trivial tree G has at least two vertices of degree 1 .
Proof:
$d(v) \geq 1$ for all points of v. Since G is non-trivial.

Also $\sum d(v) = 2q = 2(p-1) = 2p-2$.

Hence $d(v) = 1$ for at least two vertices.
∴ Every non-trivial tree G has at least two vertices of degree 1.

UNIT II

TREES, CONNECTIVITY & PLANARITY

Spanning trees – Fundamental circuits – Spanning trees in a weighted graph – cut sets – Properties of cut set – All cut sets – Fundamental circuits and cut sets – Connectivity and separability – Network flows – 1-Isomorphism – 2-Isomorphism – Combinational and geometric graphs – Planer graphs – Different representation of a planer graph.

PART-A

1.Define Minimum Spanning Tree:

SOLUTION:

In a weighted graph, a minimum spanning tree is a spanning tree that has minimum weight that all other spanning trees of the same graph. In real world situations, this weight can be measured as distance, congestion, traffic load or any arbitrary value denoted to the edges.

2. Define Spanning Tree:

SOLUTION:

A spanning tree is a subset of Graph G, which has all the vertices covered with minimum possible number of edges. Hence, a spanning tree does not have cycles and it cannot be disconnected.

3. **Define the Terms: Rank and Nullity**

SOLUTION:

Rank refers to the number of branches in any spanning tree G.r = n-k

Nullity refers to the number of Chords in G.

4. What is a fundamental circuit?

SOLUTION:

A circuit formed by adding a chord to a spanning tree T, is called a fundamental circuit.

5. Define a cutset.

SOLUTION:

A cut set of a connected graph G is a set S of edges with the following properties:

* The removal of all edges in S disconnects G.

* The removal of some (but not all) of edges in S does not disconnects G.

6. When a graph is said to be bipartite graph?

SOLUTION:

◦ if there exists a way to partition the set of vertices V, in the graph into two sets V1 and V2.

◦ where $V1 \cup V2 = V$ and $V1 \cap V2 = \varnothing$, such that each edge in E contains one vertex from V1 and the other vertex from V2.

7. What is Vertex connectivity?

SOLUTION:

The connectivity (or vertex connectivity) K (G) of a connected graph G (other than a complete graph) is the minimum number of vertices whose removal disconnects G. When $K(G) \geq k$, the graph is said to be k connected (or k-vertex connected). When a vertex is removed, all the edges incident to it are also removed.

8. Define articulation point

SOLUTION:

A vertex in an undirected connected graph is an articulation point (or cut vertex) iff removing it (and edges through it) disconnects the graph. Articulation points represent vulnerabilities in a connected network – single points whose failure would split the network into 2 or more disconnected components. They are useful for designing reliable networks.

9. Define radius and diameter.

SOLUTION:

The eccentricity of a centre in a tree is defined as the radius of the tree.

The diameter of the tree T, on the other hand is defined as the length of the longest path T.

10.Define Binary tree.

SOLUTION:

A binary tree is defined as the tree in which there is exactly one vertex of degree two, and each of the remaining vertices is of degree one or three.

11. Define Pendant Vertex.

SOLUTION:

A leaf vertex (also pendant vertex) is a vertex with degree one.

12. Define Kruskal's algorithm:

SOLUTION:

Start with *no* nodes or edges in the spanning tree, and repeatedly add the cheapest edge that does not create a cycle.

13. Define Prim's algorithm:

SOLUTION:

Start with any one node in the spanning tree, and repeatedly add the cheapest edge, and the node it leads to, for which the node is not already in the spanning tree.

14. Define weighted graph.

SOLUTION:

A weighted graph is a graph for which each edge has an associated real number, called the weight of the edge. The sum of the weights of all of the edges is the total weight of the graph.

15.Define network flow.

SOLUTION:

A network flow graph G = (V,E) is a directed graph with two special vertices: the source vertex s, and the sink (destination) vertex t.

16. What is Branch and Chord?

SOLUTION:

An edge in a spanning tree T is called a Branch of T.

An edge of G that is not in a given spanning tree T is called a Chord.

17. What is elementary tree transformation?

SOLUTION:

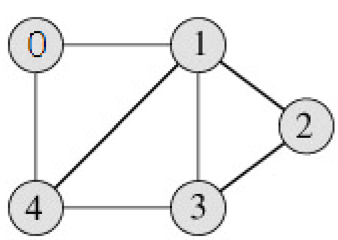
The generation of one spanning tree to another through addition of a chord and deletion of an appropriate branch is called a cyclic interchange or element tree transformation.

18.Define bridge.

SOLUTION:

An edge in an undirected connected graph is a bridge iff removing it disconnects the graph. For a disconnected undirected graph, definition is similar, a bridge is an edge removing which increases number of connected components.

19. For the following graph given below draw the adjacency matrix



SOLUTION:

The adjacency matrix for the above example graph is:

$$\begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 1 \\ 4 & 1 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

21.What is a block and explain with a diagram?

SOLUTION:

A block of a graph is a maximal connected subgraph with no cut vertex – a subgraph with as many edges as possible and no cut vertex.

PART-B

1. **Write about Spanning trees with relevant theorems?**
   SOLUTION:

A spanning tree is a subset of Graph G, which has all the vertices covered with minimum possible number of edges. Hence, a spanning tree does not have cycles and it cannot be disconnected

Rank refers to the number of branches in any spanning tree G.$r = n - k$

Nullity refers to the number of Chords in G.

A circuit formed by adding a chord to a spanning tree T, is called a fundamental circuit.

A cut set of a connected graph G is a set S of edges with the following properties:

* The removal of all edges in S disconnects G.
* The removal of some (but not all) of edges in S does not disconnects G.

THEOREM: 1

Statement: Every connected graph has at least one spanning tree.

Proof:

Let G be a connected graph.

If G has no circuits, then G is tree and G itself is a spanning tree of G.

If G has circuits, delete an edge from each cycle, then G_1 is circuit free connected and contains all vertices.

This graph $G_1$is a spanning tree of G. Thus, G has a spanning tree.

Hence, Every connected graph has at least one spanning tree.

THEOREM: 2

Statement: **A** Hamilton path is a spanning tree.

Proof :

Let T be a Hamilton path of a graph G.

Clearly T contains all the vertices of G, which traverses through every vertex and hence every edge only once.

Hence T is connected, circuit less graph having all the vertices of G.

∴ T is a spanning tree.

Hence, **A** Hamilton path is a spanning tree.

2 .**Explain Spanning trees in a Weighted Graph.**

SOLUTION:

If G is a weighted graph, then the weight of a spanning tree T of G is defined as the sum of the weights of all branches in T.

A spanning tree with the smallest weight in a weighted graph is called a shortest spanning tree.

THEOREM: Necessary and sufficient condition for a spanning tree to be shortest.

STATEMENT:A spanning tree T is a shortest spanning tree if and only if there exists no other spanning tree at a distance of one from T whose weight is smaller than that of T.

Proof:

Given: Let T be a spanning tree in G.

$\Longrightarrow$ No spanning tree at a distance of one from T_1 whose weight is shorter than that of T_1

Suppose that T_2 is a spanning tree in G.

$\Longrightarrow$ the weight of $T_1 = T_2$

T_2 must satisfy the hypothesis of the theorem.

Consider an edge in T_2 which is not in T_1.

Adding e to T_1forms a fundamental circuit with branches in T_1.

Some, but not all of the branches in T_1 that form the fundamental circuit with e may also be in T_2; each of these branches in T_1has a weight smaller than or equal to that of e, because of the assumption on T_1.

Almost all those edges in this circuit which are not in T_2 at least one, say b_j must form some fundamental circuit containing e. Because of the minimality assumption on T_2 weight of b_j cannot be less than that of e.

$\therefore b_j$ must have the same weight as e.

Hence the spanning tree$T_1{}' = (T_1 \cup e - b_j)$, obtained from T_1through one cycle exchange has the same weight at T_1.

But T_1has one edge more in common with T_2, and it satisfies the condition of the theorem.

This argument can be repeated, producing a series of trees of equal weight, T_1, each a unit distance closer to T_2, until we get T_2 itself.

This proves that if none of the spanning trees at a unit distance from T is shorter than T, no spanning tree shorter than T exists in the graph.

Hence, A spanning tree T is a shortest spanning tree if and only if there exists no other spanning tree at a distance of one from T whose weight is smaller than that of T.

3. **Explain All Cut Sets in a Graph and give Theorem proof with an example.**

SOLUTION:

A cut set of a connected graph G is a set S of edges with the following properties:

* The removal of all edges in S disconnects G.

* The removal of some (but not all) of edges in S does not disconnects G.

PROPERTIES OF CUTSET:

(1) Every cut set in a connected graph must contain at least one branch of every spanning tree of G.
(2) In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cutset.
(3) Every circuit has an even number of edges in common with any cutset.

THEOREM:

Every cut set in a connected graph must contain at least one branch of every spanning tree of G.

Proof:

Let S be a cut set of G.

Let T be the spanning tree of G.

Suppose S does not contain any branch of T.

Then all the edges of T present in G – S.

Therefore S is not a cut set.

Hence a cut set in a connected graph must contain at least one branch of every spanning tree of G.

4. **Write about Fundamental circuits and cut sets with theorem proof?**

SOLUTION:

Consider a spanning tree T in a connected graph. Adding any one chord of T will create exactly circuit. Such a circuit formed by adding a chord to a spanning tree, is called a fundamental circuit.

The cutest is a fundamental cutset if it contains exactly one edge of the spanning tree T.

If we want to find the fundamental circuits, a spanning tree is essential because a spanning tree can give the set of fundamental circuits. Similarly we can get a set of fundamental cut-set, a spanning tree is essential.

THEOREM:

With respect to given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with chords in S and in no others.

Proof:

Let a fundamental cut-set S determined a branch b_i be $S = \{b_1, c_1, c_2, c_3, \dots \dots c_p\}$,

Let $\Gamma_1$be the fundamental circuit determined by a chord c_i, $\Gamma_1 = \{c_1, b_1, b_2, b_3, \dots \dots b_q\}$.

Since, the number of edges common to S and Γ_1 must be even, b_i must be in Γ_1.

The same is true for the fundamental circuits made by chords $c_1, c_2, c_3, \dots \dots c_p$.

On the other hand , suppose that b_i occurs in a fundamental circuits Γ_{p+1} made by a chord other than $c_1, c_2, c_3, \dots \dots c_p$.

Since, none of the chords $c_1, c_2, c_3, \dots \dots c_p$ in Γ_{p+1}, there is only one edge b_i common to a circuit Γ_{p+1} and the cut set S , which is not possible.

Hence, With respect to given spanning tree T, a branch b_i that determines a fundamental cut-set S is contained in every fundamental circuit associated with chords in S and in no others.

5. **Explain Connectivity and Separability?**

SOLUTION:

Cut set has Connectivity and Separability

(i) Edge connectivity
(ii) Vertex connectivity
(iii) Separability

Edge connectivity

(i) In an connected graph, the number of edges in the smallest cut set is defined as the edge connectivity of G.
(ii) It is usually defined by k(G) or k(e).
(iii) Equivalently edge connectivity is defined as the minimal number of edges whose removal reduces the rank of the graph by one.

Vertex connectivity

(i) The vertex connectivity of a connected graph is defined as the minimal number of vertices whose removal from G leaves the remaining subgraph disconnected. This is denoted by k(v).

Separability

(i) A connectivity graph is said to be separable if its vertex connectivity is one. A connected graph which is not separable is termed as non-separable graph.

6. **Explain 1-Isomorphism with theorem proof?**

SOLUTION:

Two Graphs G_1 and G_2 are said to be 1 – isomorphic if they become isomorphic to each other when we repeatedly “decompose”, a cut vertex into two vertices to produce two disjoint sub graphs.

1. Two non-separable graphs are 1 – isomorphic iff they have isomorphic to each other.
2. A non-separable graph is one block; 1-isomorphism for nonseparable graphs is same as isomorphism.
3. For separable graph 1- isomorphism is different from from isomorphism.
4. Graphs that are isomorphic are also 1 – isomorphic; but 1 – isomorphic graph may not be isomorphic.

THEOREM:

If G_1 and G_2 are two 1 – isomorphic graphs, the rank of G_1 equals the rank of G_2 and the nullity of G_1 Equals the nullity of G_2.

PROOF:

Whenever a cut vertex in a graph G is split into two vertices, the number of components in G increases by one.

Therefore, the rank of G equals the number of vertices of G – number of components in G .

Two 1 – isomorphic graphs have the same number of edges since the edges are destroyed or new edges created by operation 1.

Two graphs with equal rank and with equal number of edges must have the same nullity.

Nullity = Number of edges – rank.

7. **Explain 2-Isomorphism with examples?**

SOLUTION:

DEFINITION: 2-Isomorphism

Two graphs G_1 and G_2 are said to be 2 – isomorphic if they become isomorphic after undergoing operation 1 or operation 2 or both any number of times.

DEFINITION: Operation – 1

"Decompose " a cut-vertex into two vertices to produce two disjoint sub graphs.

DEFINITION: Operation – 2

Merge the vertices so as to rejoin the sub graphs.

FORMAL DEFINITION

Two graphs G_1 and G_2 are said to be 2 – isomorphic if they become isomorphic after undergoing

(i) Normal split on a single cut vertex (or)
(ii) Operations 1 and 2 as mentioned above (or)
(iii) Both of the above operations for a number of times.

* The rank or nullity of the graph remain unchanged after applying the operations 1 and 2 as no edges are newly added or destroyed.

* 1 – isomorphic graphs are 2 – isomorphic.

8. **a. Write Short notes on Network flows?**

SOLUTION:

1. In networks of telephone lines, highways, railroads, pipelines of oil (or gas or water), etc., the maximum rate of flow from one station to another is an important entity.
2. Such networks are represented by a weighted connected graph in which the vertices are the stations and the edges are lines through which the given commodity (oil,gas,water,no of messages, no of cars, etc.,) flows.
3. The weight (a real positive number) associated with each edge represents the capacity of the line, i.e., the maximum amount of the flow possible per unit time.
4. Following assumptions are to be made in such networks.
   (i) The total rate of commodity entering is equal to the rate leaving at each intermediate vertex.
   (ii) There is no accumulation or generation of the commodity at any vertex.
   (iii) The flow through a vertex is limited only by the capacities of edges incident on it.
   (iv) The lines are lossless.

8.b. Write about combinational and geometric Graphs?

SOLUTION:

PLANAR GRAPH

1. A planar graph is a graph that can be embedded in the plane.
2. It can be drawn on the plane in such a way that its edges intersect only at their endpoints.
3. It can be drawn in such a way that no edges cross each other.
4. It is also called as plane graph or planar embedding of the graph.
5. Non planar graphs are those that cannot be drawn on a plane without intersection of edges.

FORMAL DEFINITION

A graph is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two edges intersect each other.

EMBEDDING

1. Representing a graph geometrically on any surface such that the edges never intersect is called embedding.
2. Embedding of a planar graph on a plane surface is called a plane representation of the graph.

NON – PLANAR GRAPH

A graph that cannot be drawn on a plane without a cross over between its edges is called non-planar.

1. A geometric representation of a graph without crossover of its edges is called a plane embedding of the graph.
2. An embedding of a planar graph is called a plane graph.

9. **Write in detail about different representation of a planar graph?**

SOLUTION:

Phases

A planar representation of a graph divides the plane in to a number of connected regions, called faces, each bounded by edges of the graph.For every graph G, we denote n (G) the number of vertices, e (G) the number of edges, f (G) the number of faces.

Degree

The degree of a face d (f) is the number of edges bounding the face f.

Theorem :

A graph is embeddable in the sphere if and only if it is embeddable in the plane.

Proof.:

We show this by using a mapping known as stereographic projection. Consider a spherical surface S touching a plane P at the point SP (called South Pole). The point NP (called the point of projection or North Pole) is on S and diametrically opposite SP. Any point z on P can be projected uniquely onto S at z' by making NP, z and z' collinear. In this way any graph embedded in P can be projected onto S. Conversely, we can project any graph embedded in s onto P , choosing NP so as not to lie an any vertex or edge of the graph.

Theorem (Euler's formula) :

If G is a connected planar graph, for any embedding G' the following formula holds:

n (G) +f (G) = e (G) +2

Proof.:

By induction on f.

· For f (G) = 1, G is a tree. For every tree, e (G) = n (G) -1, so n (G) +1= e (G) + 2

n (G) + f(G) = e(G) + 2 and the formula holds.

· Suppose it holds for all planar graphs with less than f faces and suppose that G' has f ³ 2 faces.

· Let (u, v) be an edge of G which is not a cut-edge. Such an edge must exists because G' has more than one face. The removal of (u.v) will cause the two faces separated by (u, v) to combine, forming a single face.

Hence (G-(u, v))' is a planar embedding of a connected graph with one less face than G', hence:

f (G - (u,v)) = f(G) - 1

n (G - (u,v) = n(G)

e (G - (u,v)) = e(G) - 1

But by the induction hypothesis:

n (G- (u,v)) + f(G - (u,v)) = e(G - (u,v)) + 2

and so, by substitution:

n (G) + f(G) = e(G) + 2

Hence, by induction, Euler's formula holds for all connected planar graphs.

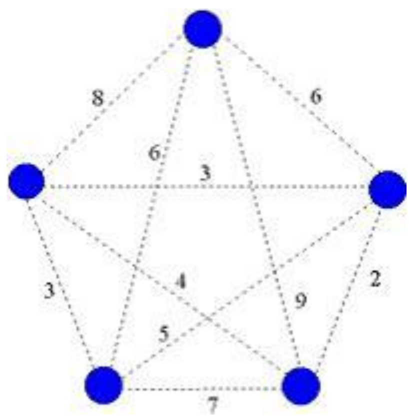
10 .**Evaluate using prim's algorithm**

SOLUTION:

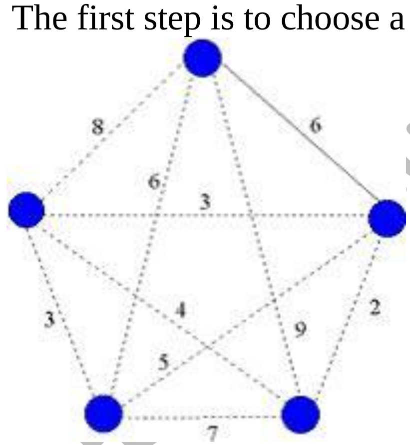
Prim's algorithm is an alternative way of finding a minimum spanning tree for a network G. Again, we start with a graph H, consisting of all the vertices of G, but with no edges. Then we follow the following instructions:

1. Pick any vertex v in the graph
2. Make it the only member of a set S
3. Consider the set E of edges joining one member of S to a vertex not in S.
4. Find the member of E with the lowest weight. If there are more than one such member, pick any one.
5. Add this edge to H, and the add the vertex not in S, to S.
6. If H is not connected, then go back to step 3. Otherwise, H is a minimum spanning tree of G.

EXAMPLE:

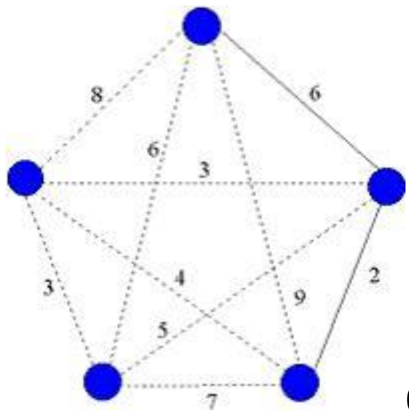


The first step is to choose a vertex to start with. This will be the top most vertex.



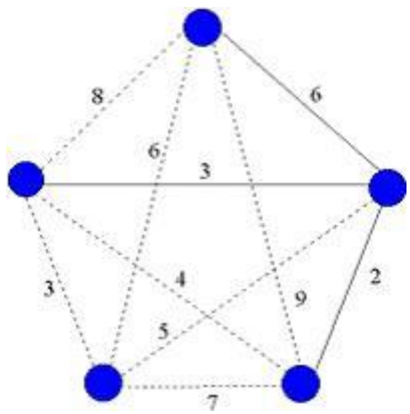
(b)

At the start, only the top vertex is in S. The edges with lowest weight between S, and other vertices are those with weight 6, so in figure (b) one is chosen, thus adding the middle right vertex to S.



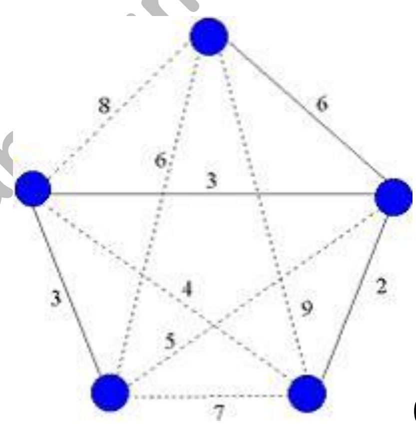
(c)

In figure (c) the edge of weight 2 is added, adding the bottom right vertex to S.



(d)

In figure (d), one of the edges of weight 3 is added, the other not having a vertex in S on either end.

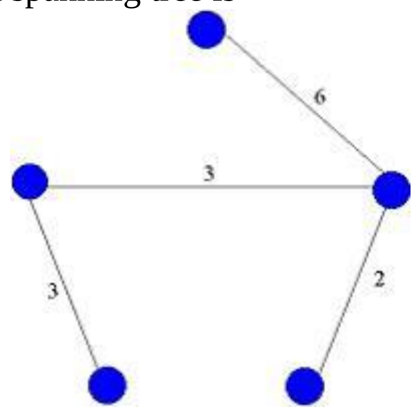


(e)

This edge is then added in figure (e). This is now allowed, as the middle left vertex was added to S in figure (d).

The total weight is $2 + 3 + 3 + 6 = 14$

The final spanning tree is



(f)

UNIT III

MATRICES, COLOURING AND DIRECTED GRAPH

Chromatic number – Chromatic partitioning –Chromatic polynomial – Matching – Covering – Four color problem – Directed graphs – Types of directed graphs – Digraphs and binary relations – Directed paths and connectedness – Euler graphs

PART-A

1. **Define Chromatic Number.**

The chromatic number of a graph G is the smallest number of colors needed to color the vertices of so that no two adjacent vertices share the same color.

2. **What is proper colouring?**

Painting all the vertices of a graph with colours such that no two adjacent vertices have the same color is called proper colouring.

3. **What is colouring problem?**

Proper colouring of a graph with minimal number of colour is called colouring problem.

4. **What is chromatic polynomial?**

A properly coloured graph G with n-vertices (many ways) and using large number of colours expressed by means of a polynomial, such a polynomial is called chromatic polynomial.

5. **Write about finding all maximal independent sets?**

A maximal independent set is an set to which no other vertex can be added without destroying its independence property.

6. **Define chromatic partioning.**

A proper coloring of a graph naturally induces a partitioning of the vertices into different subsets.

7 .**What is four colour conjectures?**

Any map (planar graph) can be properly coloured with four colours.

8 .**What is dominating sets?**

A dominating set (or externally stable set) in a graph G is a set of vertices that dominates every vertex v in G in the following sense. Either v is included in the dominating set or is adjacent to one or more vertices included in the dominating set.

9 .**Write about minimal dominating sets?**

A minimal dominating set is a dominating set from which no vertex can be removed without destroying its dominance property.

10 .**Define Isomorphic digraphs:**

Two digraphs are said to be isomorphic if their underlying graphs are isomorphic and the direction of the corresponding arcs are same.

11. **What is a maximal matching?**

A maximal matching is a matching to which no edge in the graph can be added.

Example: in a complete graph of three vertices (triangle) any single edge is a maximal matching.

12. **What is an edge covering?**

A set of edges that covers a graph G is said to be an edge covering, a covering sub graph or simply a covering of G.

13. **What is Equivalence Relation?**

A binary relation is said to be an equivalence relation if it is reflexive symmetric and transitive.

Examples: “is parallel to”, “is equal to”, “is isomorphic to”.

14. **Define digraphs**

A directed graph (or digraph) is a pair (V,E), where V is a non empty set and E is a set of ordered pairs of elements taken from the set V.

15. **Name the types of Simple Digraphs:**

A digraph that has no self-loop or parallel edges is called a simple digraph.

Asymmetric Digraphs:

Digraphs that have at most one directed edge between a pair of vertices, but are allowed to have self-loops, are called asymmetric or antisymmetric.

Symmetric Digraphs:

Digraphs in which for every edge (a, b) (i.e., from vertex a to b) there is also an edge (b, a).

16. **Write notes on complete digraphs.**

A complete undirected graph was defined as a simple graph in which every vertex is joined to every other vertex exactly by one edge.

For digraphs we have two types of complete graphs.

A **complete symmetric digraph** is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex, and a complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge between every pair of vertices.

A **complete asymmetric digraph** of n vertices contains n (n - 1)/2 edges, but a complete symmetric digraph of n vertices contains n(n - 1) edges. A complete asymmetric digraph is also called a tournament or a complete tournament.

17 .**Define a Euler Graph.**

A closed walk in a graph G containing all the edges of G is called an Euler line in G. A graph containing an Euler line is called an Euler graph.

18. **A connected graph G is an Euler graph if and only if all vertices of G are of even degree.**

Proof:

Let G (V, E) be an Euler graph.

Thus G contains an Euler line Z, which is a closed walk. Let this walk start and end at the vertex u ∈ V. Since each visit of Z to an intermediate vertex v of Z contributes two to the degree of v and since Z traverses each edge exactly once, d (v) is even for every such vertex..Each intermediate visit to u contributes two to the degree of u, and also the initial and final edges of Z contribute one each to the degree of u. So the degree d(u) of u is also even.

19. **Define Arborescence**

A digraph G is said to be an arborescence if,

i) G contains neither circuit – neither directed nor semi circuit.

ii) In G there is precisely one vertex v of zero – in degree.

20. **Write notes on strongly connected and weakly connected digraphs.**

A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.

A digraph G is said to be weakly connected if its corresponding undirected graph is connected but G is not strongly connected.

PART-B

1 Explain Chromatic number with necessary proof.

SOLUTION:

The **chromatic number** of a graph is the smallest **number** of colors needed to color the vertices of so that no two adjacent vertices share the same color (i.e.,) the smallest value of possible to obtain a k-coloring.

1. Kn: The complete graph on n vertices has chromatic number n. To see that it is at least n, simply paint each of the vertices {v1, . . . vn} of V (Kn) a different color (say, vi is painted i;) then every edge trivially has two endpoints of different colors. To see that this is necessary, take any proper coloring of Kn, and look at any vertex vi: because it’s connected to every other vertex, it cannot be the same color as any other vertex (and therefore must have a different color than every other vertex, which forces n colors.)

2. Edgeless graphs: If a graph G has no edges, its chromatic number is 1; just color every vertex the same color. These are also the only graphs with chromatic number 1; any

graph with an edge needs at least two colors to properly color it, as both endpoints
of that edge cannot be the same color.

3. Bipartite graphs: By definition, every bipartite graph with at least one edge has chromatic number 2.

4. The pentagon: The pentagon is an odd cycle, which we showed was not bipartite; so its chromatic number must be greater than 2. In fact, its chromatic number is 3:

K- COLORABLE

We say that a graph G is k-colorable if we can assign the colors1 {1, . . . k} to the vertices in V (G), in such a way that every vertex gets exactly one color and no edge in E(G) has both of its endpoints colored the same color.

We call such a coloring a proper coloring, though sometimes where it's clear what we mean we'll just call it a coloring.

Alternately, such graphs are sometimes called k-partite.

For a fixed graph G, if k is the smallest number such that G admits a k-coloring, we say that the chromatic number of G is k, and write χ (G) = k.

Definition.

For a graph G, let Δ(G) denote the maximum degree of any of G's vertices, and δ(G) denote the minimum degree of any of G's vertices.

THEOREM

For any graph G, χ(G) ≤ Δ(G) + 1.

Proof. The algorithm here is remarkably simple, but at the same time important enough that we give it a name: the greedy algorithm.

We define it here:

• (Greedy algorithm.) As input: take in a graph G with vertex set V (G) = {v1, . . . vn}, and a list of potential colors N.

• At stage k: look at v_k and color it the smallest color in N not yet used on any of v_k's neighbours.

By construction, this creates a proper coloring of G. As well, because each vertex has ≤ Δ(G) neighbors, we'll always have at least one choice of a color that's less than Δ(G)+ 1; therefore, this creates a proper coloring of G that uses ≤ Δ(G) + 1 colors! So χ (G) ≤ Δ (G) + 1, as claimed.

To sum up: we've shown that for any graph G, we have
ω(G) ≤ χ(G) ≤ Δ(G) + 1.

2 Explain Chromatic Partitioning in detail.

SOLUTION:

A set S c V is a Pk-set if x((S)) s k, where (S) is the subgraph of G induced by S. A partition {$VI, V2, l .., Vn$} of V is a. Pk-partition if each I$ is a P,-set. A &-coloring of G is a coloring of the vertices Z such that the set of all vertices receiving the same color is a Pk-set. A Pk-coloring which uses r colors is called a (k, r)-coloring. If there exists a (k, r)-coloring of G for some r s n, then G is said to be (k, n)-colorable.

The chromatic partition number $x\sim(G)$ of G is the minimum number of colors needed in a P,-coloring of G. If $xk(G)$ = n, then G is said to be (k, n)-chromatic. Clear, xl(G) = x(G) and xk(G) = 1 for all k ax(G). Thus, G is any bipartite graph, %JG) = 1 for all n 3 2, and for an odd cycle Cv, x&Z,,) = 2 and x&J = 1, for all n 2 3. For any graph G, xk(G) s Xi(G) when j s k.

For a real number r, let [r] and {r} respectively denote the greatest integer not exceeding r, and the least integer not less than r.

3. Explain Chromatic Polynomial with Theorem proof.

SOLUTION:

Chromatic Polynomials of a graph

This is a special function that describes the number of ways we can achieve a proper coloring on a graph G given k colors.

If G is a simple graph, we write PG(k) as the number of ways we can achieve a proper coloring on the vertices of G given k colors and PG is called the Chromatic Function of G. If k < χ (G), then PG(k) = 0. If we want to color the null graph N3 with k colors, we notice that this can be done k3 ways because there are k color options for each vertex since no vertex is adjacent to another . In general, we know that PNn(k) = k^n.

For a graph *G* with *n* vertices, the chromatic polynomial is defined as the unique interpolating polynomial of degree at most *n* through the points

If *G* does not contain any vertex with a self-loop, then the chromatic polynomial is a monic polynomial of degree exactly *n*. In fact for the above example we have:

The chromatic polynomial includes at least as much information about the colorability of *G* as does the chromatic number. Indeed, the chromatic number is the smallest positive integer that is not a zero of the chromatic polynomial,

4 Write in detail about Matching in a graph?

SOLUTION:

Definition:

A matching of graph G is a sub graph of G such that every edge shares no vertex with any other edge. That is, each vertex in matching M has degree one.

Definition :

The size of a matching is the number of edges in that matching.

Definition :

A matching is maximum when it has the largest possible size.

Definition :

The matching number of a graph is the size of a maximum matching of that graph.

Definition :

A matching of a graph G is complete if it contains all of G's vertices. Sometimes this is also called a perfect matching.

Hall's Marriage Theorem.

Philip Hall in 1935 gave us the condition for when a complete matching is possible in a bipartite graph. An easy was to visualize this is to consider the following situation: Suppose we are pairing up N boys and N girls (if they were not both N then clearly there is no way for a matching of our bipartite graph to be complete). Now each girl comes up with a list of acceptable mates that she likes, some subset of the N boys. Since these boys are of the gentlemanly type, none of them will reject a proposal if given to them. This situation can be represented by a bipartite graph, where an edge represents the event that a specific girl likes a specific guy.

The **assignment problem** is one of the fundamental combinatorial optimization problems in the branch of optimization or operations research in mathematics. It consists of finding a maximum weight matching (or minimum weight perfect matching) in a weighted bipartite graph.

5 Explain in detail about Coverings?

SOLUTION:

Covering

A covering graph is a sub graph which contains either all the vertices or all the edges corresponding to some other graph.

Edge covering.

A sub graph which contains all the vertices is called a **line/edge covering**.

Vertex covering.

A sub graph which contains all the edges is called a **vertex covering**.

Line Covering

Let G = (V, E) be a graph. A subset C(E) is called a line covering of G if every vertex of G is incident with at least one edge in C, i.e.,deg(V) ≥ 1 ∀ V ∈ G because each vertex is connected with another vertex by an edge. Hence it has a minimum degree of 1.

Minimum Line Covering

It is also known as **Smallest Minimal Line Covering**. A minimal line covering with minimum number of edges is called a minimum line covering of 'G'. The number of edges in a minimum line covering in 'G' is called the **line covering number** of 'G' (α_1).

- Every line covering contains a minimal line covering.
- Every line covering does not contain a minimum line covering (C_3 does not contain any minimum line covering.
- No minimal line covering contains a cycle.
- If a line covering 'C' contains no paths of length 3 or more, then 'C' is a minimal line covering because all the components of 'C' are star graph and from a star graph, no edge can be deleted.

Vertex Covering

Let 'G' = (V, E) be a graph. A subset K of V is called a vertex covering of 'G', if every edge of 'G' is incident with or covered by a vertex in 'K'.

Minimal Vertex Covering

A vertex 'K' of graph 'G' is said to be minimal vertex covering if no vertex can be deleted from 'K'.

Minimum Vertex Covering

It is also known as the smallest minimal vertex covering. A minimal vertex covering of graph 'G' with minimum number of vertices is called the minimum vertex covering.

The number of vertices in a minimum vertex covering of 'G' is called the vertex covering number of G (α_2).

6. Explain four color problems with theorem proof.

SOLUTION:

The Four Colour Theorem is famous for being the first long-standing mathematical problem to be resolved using a computer program.

Four Colour Theorem: The regions of any simple planar map can be coloured with only four colours, in such a way that any two adjacent regions have different colours.

Definition: A planar map is a set of pairwise disjoint subsets of the plane, called regions. A simple map is one whose regions are connected open sets.

Definition: Two regions of a map are adjacent if their respective closures have a common point that is not a corner of the map.

Definition: A point is a corner of a map iff it belongs to the closures of at least three regions.

FIRST STEP: The first step in the proof of the Four-Color Theorem consists precisely in getting rid of the topology, reducing an infinite problem in analysis to a finite problem in combinatorics. This is usually done by constructing the dual graph of the map, and then appealing to the compactness theorem of propositional logic. However, as we shall see below, the graph construction is neither necessary nor sufficient to fully reduce the problem to combinatorics. Therefore, we'll simply restrict the rest of this outline to connected finite maps whose regions are finite polygons and which are bridgeless: every edge belongs to exactly two polygons. Every such polyhedral map satisfies the Euler formula

$N - E + F = 2$

where N, E, and F are respectively the number of vertices (nodes), sides (edges), and regions (faces) in the map.

SECOND STEP: The next step consists in further reducing to cubic maps, where each node is incident to exactly three edges, by covering each node with a small polygon.

In a cubic map we have $3N = 2E$, which combined with the Euler formula gives us that the average number of sides (or arity) of a face is $2E/F = 6 - 12/F$.

The proof proved by induction on the size of the map.

Hence four color problem is solved.

7.State and prove five color theorem.

SOLUTION:

FIVE COLOR THEOREM:

The five color theorem is implied by the stronger four color theorem, but is considerably easier to prove

STATEMENT:

The vertices of every planar graph can be properly colored with five colors.

This theorem can be proved by principle of mathematical induction.

5-color theorem – Every planar graph is 5-colorable.

Proof:

Proof by contradiction.

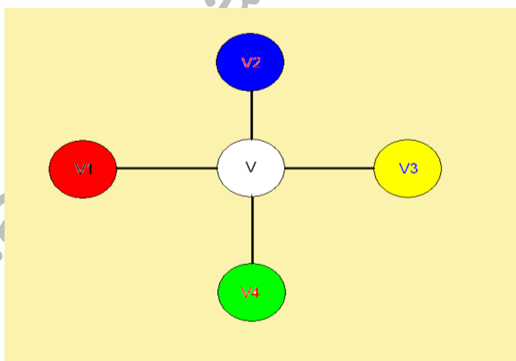
Let G be the smallest planar graph (in terms of number of vertices) that cannot be colored with five colors.

Let v be a vertex in G that has the maximum degree. We know that deg(v) < 6 (from the corollary to Euler's formula).

Case1: deg(v) ≤ 4. G-v can be colored with five colors.

There are at most 4 colors that have been used on the neighbors of v. There is at least one color then available for v.

So G can be colored with five colors, a contradiction.

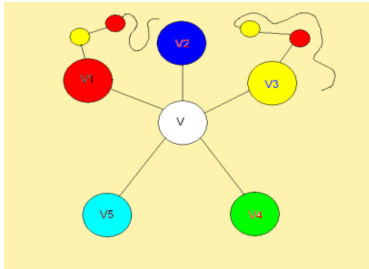


Case 2: deg(v) = 5. G-v can be colored with 5 colors.

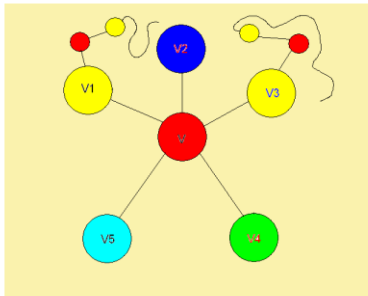
If two of the neighbors of v are colored with the same color, then there is a color available for v.

So we may assume that all the vertices that are adjacent to v are colored with colors 1,2,3,4,5 in the clockwise order.

Consider all the vertices being colored with colors 1 and 3 (and all the edges among them).

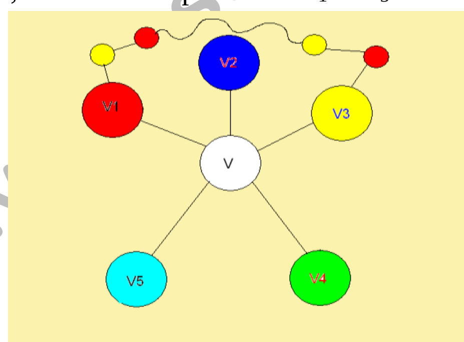


If this sub graph G is disconnected and v_1 and v_3 are in different components, then we can switch the



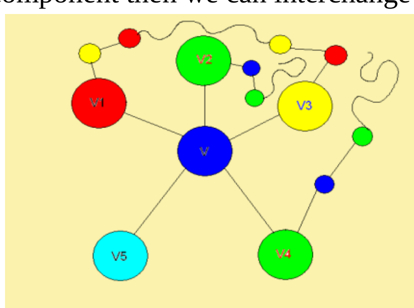
colors 1 and 3 in the component with v_1.

This will still be a 5-coloring of G-v. Furthermore, v_1 is colored with color 3 in this new 5-coloring and v_3 is still colored with color 3. Color 1 would be available for v, a contradiction. Therefore v_1 and v_3 must be in the same component in that subgraph, i.e. there is a path from v_1 to v_3 such that every vertex on this



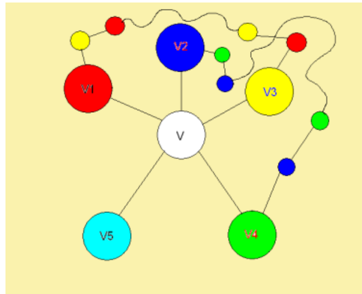
th is colored with either color 1 or color 3.

Now, consider all the vertices being colored with colors 2 and 4 (and all the edges among them). If v_2 and v_4 don't lie of the same connected component then we can interchange the colors in the chain



starting at v_2 and use left over color for v.

If they do lie on the same connected component then there is a path from v_2 to v_4 such that every vertex on that path has either color 2 or color 4.

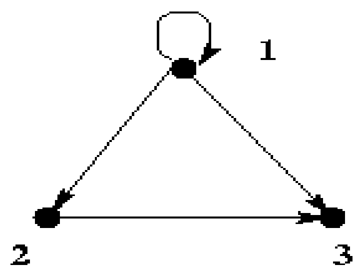


This means that there must be two edges that cross each other. This contradicts the planarity of the graph and hence concludes the proof.

8. Explain in detail about digraphs and binary relations.

SOLUTION:

A **digraph** is short for directed graph, and it is a diagram composed of points called **vertices** (nodes) and arrows called **arcs** going from a vertex to a vertex.
For example the figure below is a digraph with 3 vertices and 4 arcs.



G₁

In this figure the vertices are labeled with numbers $\mathbf{1}, \mathbf{2},$ *and* $\mathbf{3}$.
Mathematically, a digraph is defined as follows.
Definition (digraph): A **digraph** is an ordered pair of sets ***G* = (*V*, *A*)**, where ***V*** is a set of **vertices** and ***A*** is a set of ordered pairs (called **arcs**) of vertices of ***V***.
In the example, G_1, given above, ***V* = { *1, 2, 3* }**, and $A = \{ <1,1>, <1,2>, <1,3>, <2,3> \}$.

Digraph representation of binary relations

A binary relation on a set can be represented by a digraph.
Let ***R*** be a binary relation on a set ***A***, that is ***R*** is a subset of ***A*** X ***A***.
Then the digraph, call it ***G***, representing ***R*** can be constructed as follows**:**

1. The vertices of the digraph ***G*** are the elements of ***A***, and
2. $< x, y >$ is an arc of ***G*** from vertex ***x*** to vertex ***y*** if and only if $< x, y >$ is in ***R***.

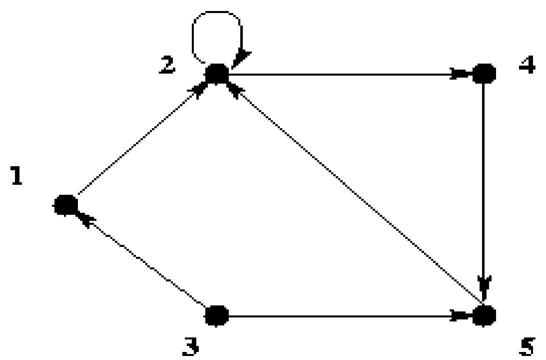
Definition (loop): An arc from a vertex to itself such as $< 1, 1 >$, is called a **loop (or self-loop)**
Definition (degree of vertex): The **in-degree** of a vertex is the number of arcs coming to the vertex, and the **out-degree** is the number of arcs going out of the vertex.
For example, the in-degree of vertex ***2*** in the digraph G_2 shown above is ***1***, and the out-degree is ***2***.
Definition (path): A **path** from a vertex x_0 to a vertex x_n in a digraph **G = (V, A)** is a sequence of vertices $x_0, x_1, \ldots, x_n$ that satisfies the following:
for each i, $0 \le i \le n-1$, $< xi, xi+1 > \epsilon A$, or $< xi+1, xi > \epsilon A$, that is, between any pair of vertices there is an arc connecting them.
A path is called a **directed path** if $< xi, xi+1 > \epsilon A$, for every i, $0 \le i \le n-1$.
If no arcs appear more than once in a path, the path is called a **simple path**. A path is called **elementary** if no vertices appear more than once in it.

Definition(connected graph): A digraph is said to be **connected** if there is a path between every pair of its vertices.

Example: In the digraph G_3 given below,

- ***1, 2, 5*** is a simple and elementary path but not directed,
- ***1, 2, 2, 5*** is a simple path but neither directed nor elementary.
- ***1, 2, 4, 5*** is a simple elementary directed path,
- ***1, 2, 4, 5, 2, 4, 5*** is a directed path but not simple (hence not elementary),
- ***1, 3, 5, 2, 1*** is a simple elementary cycle but not directed, and
- ***2, 4, 5, 2*** is a simple elementary directed cycle.

This digraph is connected.



G_3

Definition (reflexive relation): A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called **reflexive** if and only if $< a, a > \boldsymbol{\epsilon}\ \boldsymbol{R}$ for every element $\boldsymbol{a}$ of $\boldsymbol{A}$.

Example : The relation $\le$ on the set of integers $\{\boldsymbol{1, 2, 3}\}$ is $\{< \mathbf{1}, \mathbf{1} >, < \mathbf{1}, \mathbf{2} >, < \mathbf{1}, \mathbf{3} >, < \mathbf{2}, \mathbf{2} >,$ $< 2, 3 >, < \mathbf{3}, \boldsymbol{3} >\}$ and it is reflexive because $< 1, 1 >, < \mathbf{2}, \mathbf{2} >, < \mathbf{3}, \mathbf{3} >$ are in this relation. As a matter of fact $\le$ on any set of numbers is also reflexive. Similarly $\ge$ and = on any set of numbers are reflexive. However, $< OR >$ on any set of numbers is not reflexive.

Definition (irreflexive relation): A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called **irreflexive** if and only if $< a, a > \ \notin \ \boldsymbol{R}$ for every element $\boldsymbol{a}$ of $\boldsymbol{A}$.

Example : The relation > (or <) on the set of integers $\{\boldsymbol{1, 2, 3}\}$ is irreflexive. In fact it is irreflexive for any set of numbers.

Definition (symmetric relation): A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called **symmetric** if and only if for any $\boldsymbol{a}$, and $\boldsymbol{b}$ in $\boldsymbol{A}$, whenever $< a, b > \in \boldsymbol{R}$, $< b, a > \in \boldsymbol{R}$.

Example : The relation = on the set of integers $\{\boldsymbol{1, 2, 3}\}$ is $\{< 1, 1 >, < \mathbf{2}, \mathbf{2} > < \mathbf{3}, \mathbf{3} >\}$ and it is symmetric. Similarly = on any set of numbers is symmetric. However, < (or)>, $\le$ (or) $\ge$ on any set of numbers is not symmetric.

Definition (antisymmetric relation): A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called **antisymmetric** if and only if for any $\boldsymbol{a}$, and $\boldsymbol{b}$ in $\boldsymbol{A}$, whenever $< a, b > \in \ \boldsymbol{R}$, and $< b, a > \in \boldsymbol{R}$, $\boldsymbol{a} = \boldsymbol{b}$ must hold. Equivalently, $\boldsymbol{R}$ is antisymmetric if and only if whenever $< a, b > \in \ \boldsymbol{R}$, and $\boldsymbol{a} \ \neq \boldsymbol{b}$, $< b, a > \notin \ \boldsymbol{R}$ **.** Thus in an antisymmetric relation no pair of elements are related to each other.

Definition (transitive relation): A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called **transitive** if and only if for any $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in $\boldsymbol{A}$, whenever $< a, b > \in \ \boldsymbol{R}$, and $< b, c > \in \ \boldsymbol{R}$ **,** $< a, c > \in \ \boldsymbol{R}$.

Example : The relation $\le$ on the set of integers $\{\boldsymbol{1, 2, 3}\}$ is transitive, because for $< 1, 2 > \ and$ $< 2, 3 > \ in \ \le \ , \ < 1, 3 > \ is\ also\ in \ \le, for \ < 1, 1 > \ and \ < 1, 2 > \ in \ \le \ , \ < 1, 2 >$ $is\ also\ in \ \le$, and similarly for the others. As a matter of fact $\le$ on any set of numbers is also transitive. Similarly $\ge$ and = on any set of numbers are transitive.

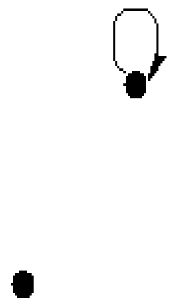
The following figures show the digraph of relations with different properties.

(a) is reflexive, antisymmetric, symmetric and transitive, but not irreflexive.

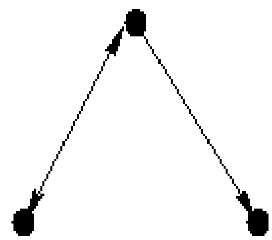
(b) is neither reflexive nor irreflexive, and it is antisymmetric, symmetric and transitive.

(c) is irreflexive but has none of the other four properties.
(d) is irreflexive, and symmetric, but none of the other three.
(e) is irreflexive, antisymmetric and transitive but neither reflexive nor symmetric.

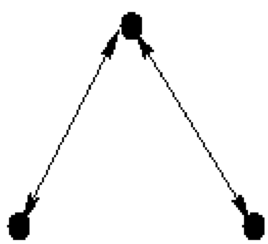
(a)



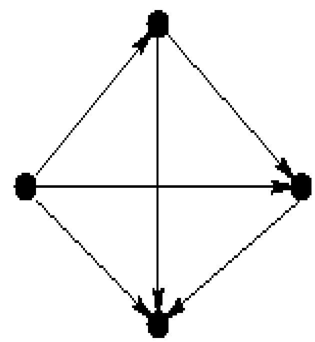
(b)



(c)



(d)



(e)

An **equivalence relation** is a relation that is reflexive, symmetric, and transitive
Example:
Equality mod m: The relation x = y (mod m) that holds when x and y have the same remainder when divided by m is an equivalence relation.

9 .Explain directed paths and Connectedness.

SOLUTION:

A path, in either an undirected or a directed graph, is a sequence of edges where the destination of each edge is the source of the next edge. For example, a sequence of edges $(u_0, u_1), (u_1, u_2),..., (u_{k-1},u_k)$ forms a path of k edges from vertex u_0 to vertex u_k. If {u, v} is an edge of an undirected graph, the path may use either (u, v) or (v, u)
The length of a path is the number of edges in it, not the number of vertices.

Connectedness in Directed Graphs

Definition: A directed graph is strongly connected if there is a path from a to b and a path from b to a whenever a and b are vertices in the graph.
Definition: A directed graph is weakly connected if there is a path between every two vertices in the

underlying undirected graph, which is the undirected graph obtained by ignoring the directions of the edges of the directed graph.
Example: G is strongly connected because there is a path between any two vertices in the directed graph. Hence, G is also weakly connected.
Definition: The subgraphs of a directed graph G that are stronglyconnected but not contained in larger strongly connected subgraphs,that is, the maximal strongly connected subgraphs, are called the strongly connected components or strong components of G.

SIMPLE DIPATH

A directed path in which no two vertices are the same (except that the initial and final vertices may be the same).

In a directed graph, vertices have both "indegrees" and "outdegrees". The indegree of a vertex is the number of arcs leading to that vertex, and the outdegree of a vertex is the number of arcs leading away from that vertex.

A vertex with an indegree of 0 is called a **source** (since one can *only* leave it) and a vertex with an outdegree of 0 is called a **sink** (since one *cannot* leave it). It is relatively easy to see that **a directed graph with no cycles has at least one source and one sink**.

- a **walk** through a directed graph is a sequence of vertices connected by arcs corresponding to the order of the vertices in the sequence;
- a "**semi-walk**" through a directed graph is a sequence of vertices in which the arc directions are ignored;
- a **path** is still a walk with no repeated vertices, and a
- "**semi-path**" is a semi-walk with no repeated vertices.

A **Hamiltonian path** (or **traceable path**) is a path in an undirected or directed graph that visits each vertex exactly once. A **Hamiltonian cycle** (or **Hamiltonian circuit**) is a Hamiltonian path that is a cycle

10. Explain Euler digraphs with theorem proof.

SOLUTION:

A directed graph is Eulerian iff every graph vertex has equal in degree and out degree. A planar bipartite graph is dual to a planar Eulerian graph and vice versa.

THEOREM:

STATEMENT:

*A **digraph** has an Euler cycle if and only if it is **connected** and the in degree of each vertex equals its out degree.*

PROOF:

For a proof we may only consider the *loop less* graphs. (A *loop* is an edge that starts and ends at the same vertex.) If a graph has an Euler cycle than obviously the number of edges that start at a vertex equals the number of edges that end there.

In the other direction, let G(V, E) be a connected digraph with V as the set of vertices and E the set of edges. If |V| = 1, i.e., there is a single vertex, there is nothing to prove.

Otherwise, start with any vertex, proceed along any outgoing edge, and continue this way until a cycle has been created. (The path can't last indefinitely since E is a finite set of edges. It could not end at a vertex from which there is no way out unless this is where the path has started.)

If the cycle contains all the edges of E of G, we are done. Otherwise, denote the edges of the cycle E', the nodes on it with the out degree (and hence the in degree) equal to 1 V', and consider the graph

G' = G (V - V', E - E'). Since G is connected, the cycle could not be a ***connected component*** (unless it's an Euler cycle, which we assumed it is not). If so, every connected component of G' shares at least one vertex with the cycle. Start with those vertices to find a cycle in each of the connected components of G'. Continue in this manner until no edges remain.

Note that the graphs G' obtained on various stages of the construction satisfy the required condition of equality of the in- degree and out degree at every vertex. This is so because that is true for G and for every vertex in the cycle.

When finished, the process ends up with a set of cycles with vertices in G whose union contains every edge from E exactly once. To conclude the proof we'll need the following proposition:

The union of a finite number of cycles of a connected digraph is a cycle.

The proof is by induction. For a single cycle, there is nothing to prove. If there are more than one, there bound to be at least two that share a vertex. Starting at this vertex we traverse first one of the cycles then the other which exactly implies that the union of the two is a cycle. We are thus able to reduce the number of cycles in the union by 1 which leads to a proof by induction on the number of cycles.

UNIT IV

PERMUTATIONS & COMBINATIONS:

Fundamental principles of counting - Permutations and combinations - Binomial theorem - combinations with repetition - Combinatorial numbers - Principle of inclusion and exclusion - Derangements - Arrangements with forbidden Positions**.**

PART-A

1 .**Define Fundamental Counting Principle:**

The principle for determining the number of ways two or more operations can be performed together.

Example: How many ways can six different books be positioned on a book shelf?

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Six different books can be positioned 720 ways on a book shelf.

2. Explain Addition rule or Sum rule.

If one task or operation can be performed in *m* ways, while a second ask can be performed in *n* ways and the two tasks cannot be done simultaneously, and then either of the tasks can be done in $m + n$ ways.

Example: There are 3 lists of computer projects consisting of 23, 15 and 19 possible projects respectively. No project is found in more than one list. How many possible projects can a student choose?

Solution: Total number of projects is 23+15+19 = 57. Since no project is found in more than one list, the number of ways a project can be chosen is 57 ways.

3. Explain Multiplication rule or Product rule.

Suppose a certain task or operation can be done in *m* ways and another task independent of the former can be done in *n* ways. Then both of them can be done in *mn* ways.

Example: There are two different collections of books consisting of 6 Mathematics books and 4 Computer Science books. In how many ways can a student select a Mathematics book and a Computer Science book?

Solution: One Mathematics book can be chosen in 6 ways and one Computer Science book can be chosen in 4 ways.

$\therefore$ Total number of ways taking one Mathematics book and one Computer Science book = (6) (4) = 24 ways.

4 .**Define Permutation.**

A permutation is an arrangement of a given collection of objects in a definite order taking some of the objects or all the objects.

5. **How many different bit strings of length 7 are there?**

Solution:

Each of the 7 places can be filled with 0 or 1.

$\therefore$ the number of bit strings of length 7 is

$2^7 = 128$

6. **Define Combination.**

A Combination is a selection of objects from a given collection of objects taken some objects or all the objects at a time. The order of selection is immaterial.

7. **Explain Inclusion – Exclusion Principle.**

A third basic principle of counting is the inclusion-exclusion principle.

If A and B are mutually exclusive, then

$n(A \cup B) = n(A) + n(B)$ or $|A \cup B| = |A| + |B|$.

If A and B are any two sets, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

If A, B, C are any three sets, then

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$.

It has applications in number theory, onto functions, derangements.

8. **Explain Pigeon Hole Principle.**

If m pigeons are assigned to n pigeonholes (m>n), then at least one pigeonhole has two or more pigeons.

9. **A salesman at a computer store would like to display 5 models of personal computers, 4 models of computer monitors and 3 models of keyboards. In how many ways can he arrange them in a row if the items of the same kind are together?**
Solution:
The total number of arrangements
= 3! X 5! X 4! X 3! = 103,680.
10. **From a college if we select 367 students, how many students have the same birthday?**
Solution: There are 367 students and the maximum number of days in a year is 366 (leap year) and so 366 possible birthdays are there. Treating the 367 students as pigeons and the 366 birthdays as pigeonholes, by Pigeonhole principle at least two students will have the same birthday.
11. **How many different 7 digits phone numbers are possible if the 1st digit cannot be a 0 or 1?**
8*10*10*10*10*10*10=8000000 ways
12. **Find the number of distinguishable permutations of the letters "MISSISSIPPI"**
11 letters with I repeated 4 times, S repeated 4 times, P repeated 2 times
11! = 39,916,800 = 34,650
4!*4!*2! 24*24*2
13. **Define Rook Polynomial.**
The rook polynomial RB(x) of a board B is the generating function for the numbers of arrangements of non-attacking rooks.

$$RB(x) = \sum_{k=0}^{\infty} r_k (B) x_k$$

where r_k is the number of ways to place k non-attacking rooks on the board. Despite the notation, this is a finite sum, since the board is finite so there is a maximum number of non-attacking rooks it can hold; indeed, there cannot be more rooks than the smaller of the number of rows and columns in the board.
14 .**What is an Inclusion Map?**
Given a subset **B** of a set **A** the injection f: B → A is defined by $f\ (b) = b$ for all b belongs to B is called the inclusion map.
15 .**From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?**
We may have (3 men and 2 women) or (4 men and 1 woman) or (5 men only).
Required number of ways = $({}_7C_3 \times {}_6C_2) + ({}_7C_4 \times {}_6C_1) + ({}_7C_5) = 756$.
16. **List all the derangements of the number 1, 2, 3, 4 and 5 where there first three numbers are 1, 2 and 3 in some order.**
When 1, 2 and 3 are in some order there are only two derangements.
23154 and 31254.
17. **In how many ways can 8 papers in an examination be arranged so that the two Mathematics papers are not consecutive?**
Since the two Mathematics papers are not to be consecutive, remove them for the time being. Arrange the remaining 6 papers. This can be done in *P* (6, 6) = 6! ways = 720 ways.
In each of these arrangements of the 6 papers there are 7 gaps. In these gaps, the 2 Mathematics papers can be arranged in *P* (7, 2) = (7) (6) = 42 ways.
∴ the total number of ways arranging all the 8 papers = 720 x 42 = 30240 ways.
18. **How many different license plates is possible if each plate contains a sequence of 3 English alphabets followed by 3 digits? (Repetition is allowed)**
There are 26 alphabets and 10 digits.
∴ there are 26 choices for each alphabet and 10 choices for each digit.
∴ all 3 alphabets can be chosen in 26 x 26 x 26 = $26C_3$ ways,
and the 3 numbers can be chosen in 10 x10 x 10 = $10C_{3}{}_3$ ways.

∴total number of possible license plates
= $26C_3$ x $10C_3$ = 17,576,000

19. **7 students enter a restaurant where each of them has one of the following:**
a cheese burger, a chicken sandwich, a egg puff or a veg puff. How many different purchases are possible? (from restaurant point of view)

n = 7, r = 4,
no: of purchases = (n+r-1) Cr = (7+4-1) C4 =120.

20. **Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?**

Number of ways of selecting 3 consonants from 7 = $_7C_3$
Number of ways of selecting 2 vowels from 4 = $_4C_2$
Number of ways of selecting 3 consonants from 7 and 2 vowels from 4 = $_7C_3 \times {}_4C_2$
= (7× 6× 53× 2× 1)× (4 × 32 ×1) =210
It means we can have 210 groups where each group contains total 5 letters *(3 consonants and 2 vowels).*

PART-B

1 **Explain the fundamental principles of counting.**
SOLUTION:

Fundamental Principle of counting: *The number of distinguishable permutations of **n** objects, of which **a** objects are identical, another **b** objects are identical, and another **c** objects are identical, and so on, is* $\frac{n!}{a!b!C!....}$

Examples:

1. Activities: roll a die and flip a coin
   There are 6 ways to roll a die and 2 ways to flip a coin.
   There are 6 • 2 = 12 ways to roll a die and flip a coin.
2. Activities: draw two cards from a standard deck of 52 cards without replacing the cards
   There are 52 ways to draw the first card.
   There are 51 ways to draw the second card.
   There are 52 • 51 = 2,652 ways to draw the two cards.

The Counting Principle also works for more than two activities.

3. Activities: a coin is tossed five times
   There are 2 ways to flip each coin.
   There are 2 • 2 • 2 • 2 •2 = 32 arrangements of heads and tails.
4. Activities: a die is rolled four times
   There are 6 ways to roll each die.
   There are 6 • 6 • 6 • 6 = 1,296 possible outcomes.

2. A total of 1232 students have taken a course in Tamil, 879 have taken a course in English and 114 have taken a course in Telugu. Further, 103 have taken courses in both Tamil and English, 23 have taken courses in both Tamil and Telugu and 14 have taken courses in both English and Telugu. If 2092 students have taken at least one of Tamil, English and Telugu, how many students have taken a course in all three languages?

SOLUTION:

Let $T \rightarrow Students\ who\ have\ taken\ a\ course\ in\ Tamil.$

$E \rightarrow Students\ who\ have\ taken\ a\ course\ in\ English.$

$R \rightarrow Students\ who\ have\ taken\ a\ course\ in\ Telugu..$

(i.e.,) $|T| = 1232,\ \ |E| = 879, |R| = 114, |T \cap E| = 103, |E \cap R| = 23$

$|E \cap R| = 14\ \ and\ |T \cup E \cup R| = 2092$

By the principle of inclusion and exclusion we get

$$|T \cup E \cup R| = |T| + |E| + |R| - |T \cap E| - |T \cap R| - |E \cap R| + |T \cap E \cap R|$$

$$\Rightarrow \quad 2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |T \cap E \cap R|$$

$$\Rightarrow \quad |T \cap E \cap R| = 7$$

Therefore, there are seven students who have taken courses in Tamil, English and Telugu.

3. Among the first 1000 positive integers: Determine the integers which are not divisible by 5, nor by 7, nor by 9.

SOLUTION:

Let A be the number of integers divisible by 5

B be the number of integers divisible by 7.

C be the number of integers divisible by 9.

$$\therefore\ |A| = \left|\frac{1000}{5}\right| = 200$$

$$|B| = \left|\frac{1000}{7}\right| = 142$$

$$|C| = \left|\frac{1000}{9}\right| = 111$$

$$|A \cap B| = \left|\frac{1000}{LCM(5,7)}\right|\left|\frac{1000}{5X7}\right| = 28$$

$$|A \cap C| = \left|\frac{1000}{LCM(5,9)}\right|\left|\frac{1000}{5X9}\right| = 22$$

$$|C \cap B| = \left|\frac{1000}{LCM(9,7)}\right|\left|\frac{1000}{9X7}\right| = 15$$

$$|A \cap B \cap C| = \left|\frac{1000}{LCM(5,7,9)}\right|\left|\frac{1000}{5X7X9}\right| = 3$$

The number of integer divisible by 5,7 and 9.

$$|A \cup B \cup C| = 200 + 142 + 111 - 28 - 22 - 15 + 3$$
$$= 391$$

The number of integers not divisible by 5, or by 7, or by 9.

$= Total\ number\ of\ integers -$
$integers\ divisible\ by$ 5,7 & 9

$= 1000 - 391$

= 609.

4. Find the coefficient of x^{18} in

$$\left(x + x^2 + x^3 + x^4 + x^5\right)(x^2 + x^3 + x^4 + \cdots..)^5.$$

SOLUTION:

WKT $(x + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + \cdots..)^5$

$$= x(1 + x^2 + x^3 + x^4)x^{10}(1 + x + x^2 + x^3 + x^4 + \cdots..)^5$$
$$= x^{11}(1 + x^2 + x^3 + x^4)(1 - x)^{-5}$$
$$= x^{11}\left[\frac{1-x^5}{1-x}\right](1 - x)^{-5}$$
$$= x^{11}(1 - x^5)(1 - x)^{-6}$$
$$= (x^{11} - x^{16})(1 - x)^{-6}$$
$$= (x^{11} - x^{16})\sum C(6 + r - 1, r)x^r$$
$$= \sum C(6 + r - 1, r)x^{11+r} - \sum C(6 + r - 1, r)x^{16+r}$$

Hence the coefficient of x^{18} is $C(6 + 7 - 1, 7) - C(6 + 2 - 1, 2)$

$$= C(12, 7) - C(7, 2)$$
$$= 792 - 21$$
$$= 771.$$

5. **Find an explicit formula for the Fibonacci numbers.**

SOLUTION:

The sequence of Fibonacci numbers satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

And satisfies the initial conditions $f_0 = 0$ & $f_1 = 1$

$(1) \Rightarrow f_n - f_{n-1} - f_{n-2} = 0$………………..(2)

Let $f_n = r^n$ be a solution of the given equation

$(2) \Rightarrow r^n - r^{n-1} - r^{n-2} = 0$

$\Rightarrow r^n\left[1 - \frac{1}{r} - \frac{1}{r^2}\right] = 0$

$\therefore$ The characteristic equation is $r^2 - r - 1 = 0$

$$r = \frac{1 \mp \sqrt{(-1)^2 - 4(-1)}}{2}$$
$$r = \frac{1 \mp \sqrt{1+4}}{2}$$

$\Rightarrow r_1 = \frac{1+\sqrt{5}}{2},\ r_2 = \frac{1-\sqrt{5}}{2}$

$\therefore$ By Theorem,

$f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$……………….(3)

$f_0 = 0 \Rightarrow f_0 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$

$\Rightarrow \alpha_1 + \alpha_2 = 0$…………………………(4)

$f_1 = 1 \Rightarrow f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$

$(1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 2$……………..(5)

$(4)X(1+\sqrt{5}) \Rightarrow (1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 0$………(6)

$(5)X1 \qquad \Rightarrow (1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 2$………(7)

$(6)-(7) \qquad \Rightarrow 2\sqrt{5}\alpha_2 = -2$

$\alpha_2 = \frac{-1}{\sqrt{5}}$

$(4) \qquad \Rightarrow \qquad \alpha_1 = \frac{1}{\sqrt{5}}$

$(3) \quad \Rightarrow \quad f_n = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$

6. **Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?**

SOLUTION:

By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By theorem, the number of ways to select the committee is

C (9, 3) x C (11, 4) = $\frac{9!}{3!6!}\frac{11!}{4!7!} = (84)(330) = 27{,}720.$

7.State and prove Binomial Theorem.

SOLUTION:

THEOREM:

According to the theorem, it is possible to expand any power of $x + y$ into a sum of the form

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots.\binom{n}{n}x^0 y^n$$

where each $\binom{n}{k}$ is a specific positive integer known as a binomial coefficient. (When an exponent is zero,the corresponding power expression is taken to be 1 and this multiplicative factor is often omitted from the term. This formula is also referred to as the **binomial formula** or the **binomial identity**.

The Binomial Expansion Theorem can be written in summation notation, where it is very compact and manageable.

$$(x+y)^n = \sum_{0}^{n} \binom{n}{k} x^{n-k} y^k$$

BINOMIAL EXPANSION EXAMPLES

$$(x \pm y)^2 = x^2 \pm 2xy + y^2$$
$$(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$$
$$(x \pm y)^4 = x^4 \pm 4x^3y + 6x^2y^2 \pm 4xy^3 + y^4$$

The *r* th term of the expansion of $(x+y)^n = \binom{n}{k} x^{n-k} y^k$

8.Write about catalan numbers.

SOLUTION:

The nth Catalan number, denoted Cn, is the number of ways of multiplying together n symbols.

Explicit formulas for C_n include

$$C_n \equiv \frac{1}{n+1}\binom{2n}{n} \quad (1)$$

$$= \frac{(2n)!}{(n+1)!\,n!} \quad (2)$$

$$= \frac{2^n (2n-1)!!}{(n+1)!} \quad (3)$$

$$= \frac{4^n \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+2)} \quad (4)$$

$$= (-1)^n 2^{2n+1} \binom{\frac{1}{2}}{n+1} \quad (5)$$

$$= \frac{1}{n}\binom{2n}{n-1} \quad (6)$$

$$= {}_2F_1(1-n, -n; 2; 1), \quad (7)$$

where $\binom{n}{k}$ is a binomial coefficient, $n!$ is a factorial, $n!!$ is a double factorial, $\Gamma(z)$ is the gamma function, and ${}_2F_1(a,b;c;z)$ is a hypergeometric function.

A recurrence relation for C_n is obtained from

$$\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2},$$

OR

$$C_{n+1} = \frac{2(2n+1)}{n+2} C_n.$$

The values of C_n for n ≤ 10 are given in Table 1. The first eleven Catalan numbers

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	2	5	14	42	132	429	1,430	4,862	16,796

9. **Explain Derangements.**
SOLUTION:
In combinatorial mathematics, a **derangement** is a permutation of the elements of a set, such that no element appears in its original position.

Out of 24 possible permutations (4!) for handing back the tests:

ABCD, **AB**DC, **A**CB**D**, **A**CDB, **A**DBC, **A**D**C**B,

BA**CD**, *BADC*, BCA**D**, *BCDA*, *BDAC*, BD**C**A,

CAB**D**, *CADB*, C**B**A**D**, C**B**DA, *CDAB*, *CDBA*,

DABC, DA**C**B, D**B**AC, D**BC**A, *DCAB*, *DCBA*.

there are only 9 derangements (shown in blue italics above). In every other permutation of this 4-member set, at least one student gets his or her own test back (shown in bold red).

10 .**Write about rook polynomial.**
SOLUTION:
In combinatorial mathematics, a **rook polynomial** is a generating polynomial of the number of ways to place non-attacking rooks on a **board** that looks like acheckerboard; that is, no two rooks may be in the same row or column. The board is any subset of the squares of a rectangular board with *m* rows and *n* columns; we think of it as the squares in which one is allowed to put a rook. The board is the ordinary chessboard if all squares are allowed and $m = n = 8$ and a chessboard of any size if all squares are allowed and $m = n$. The coefficient of x^k in the rook polynomial $R_B(x)$ is the number of ways *k* rooks, none of which attacks another, can be arranged in the squares of *B*. The rooks are arranged in such a way

that there is no pair of rooks in the same row or column. In this sense, an arrangement is the positioning of rooks on a static, immovable board; the arrangement will not be different if the board is rotated or reflected while keeping the squares stationary. The polynomial also remains the same if rows are interchanged or columns are interchanged.

Famous examples include the number of ways to place n non-attacking rooks on:

- an entire $n \times n$ chessboard, which is an elementary combinatorial problem;
- the same board with its diagonal squares forbidden; this is the derangement or "hat-check" problem;
- the same board without the squares on its diagonal and immediately above its diagonal (and without the bottom left square), which is essential in the solution of the problèm des ménages.

The **rook polynomial** $R_B(x)$ of a board B is the generating function for the numbers of arrangements

$$RB(x) = \sum_{k=0}^{\infty} rk\ (B)xk$$

where r_k is the number of ways to place k non-attacking rooks on the board. Despite the notation, this is a finite sum, since the board is finite so there is a maximum number of non-attacking rooks it can hold; indeed, there cannot be more rooks than the smaller of the number of rows and columns in the board.

A rook polynomial is a special case of one kind of matching polynomial, which is the generating function of the number of k-edge matchings in a graph.

The rook polynomial $R_{m,n}(x)$ corresponds to the complete bipartite graph $K_{m,n}$. The rook polynomial of a general board $B \subseteq B_{m,n}$ corresponds to the bipartite graph with left vertices $v_1, v_2, ..., v_m$ and right vertices $w_1, w_2, ..., w_n$ and an edge v_iw_j whenever the square (i, j) is allowed, i.e., belongs to B. Thus, the theory of rook polynomials is, in a sense, contained in that of matching polynomials.

UNIT V
GENERATING FUNCTIONS:

Generating functions - Partitions of integers - Exponential generating function – Summation operator - Recurrence relations - First order and second order – Non-homogeneous recurrence relations - Method of generating functions.

PART-A

1 **Define Generating Function.**

A generating function is a formal power series whose coefficients give the sequence.

2 **Define Recurrence Relation.**

A mathematical relationship expressing as some combination of with. When formulated as an equation to be solved, recurrence relations are known as recurrence equations, or sometimes difference equations.

3 **Define Recursive Sequence.**

A recursive sequence, also known as a recurrence sequence, is a sequence of numbers indexed by an integer and generated by solving a recurrence equation. The terms of a recursive sequences can be denoted symbolically in a number of different notations, such as, or f[ ], where is a symbol representing the sequence.

4.Find a closed form for the generating function of 0,0,1,1,1,……

Solution:

WKT $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ \ generates\ the\ sequence\ (1,1,1,\ldots\ldots)$

$$\frac{x^2}{1-x} = \sum_{n=0}^{\infty} x^{n+2} \ \ generates\ the\ sequence\ (0,0,1,11,\ldots\ldots)$$

5.What are the steps for solving a linear homogeneous recurrence relation of degree 2:

Step 1. Get the characteristic equation of the given recurrence relation.

Step 2. Solve the characteristic equation to get the two roots r1, r2. The general solution is an = α1r1n + α2r2n

Step 3. Substitute the initial conditions into the general solution to find the constants α1 and α2.

6 **Solve the recurrence relation Fn = 5F**n−1 − 6Fn−2 **where F0 = 1 and F1 = 4**

The characteristic equation of the recurrence relation is −

$x^2 - 5x + 6 = 0,$

$(x - 3)(x - 2) = 0$

Hence, the roots are $x_1 = 3\ and\ x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is

$Fn = ax1n + bx2n, Fn = a3n + b2n\ (As\ x1 = 3\ and\ x2 = 2)$

Therefore,

1 = F0 = a30 + b20 = a+b

4 = F1 = a31 + b21 = 3a+2b

Solving these two equations, we get a = 2 and b = −1

Hence, the final solution is

Fn = 2.3n + (−1) . 2n = 2.3n − 2n

7. **List some applications of Generating Functions.**

- For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50.
- For solving recurrence relations.

- For proving some of the combinatorial identities.
- For finding asymptotic formulae for terms of sequences.

8. **What is the generating function of the infinite series; 1, 1, 1, 1..?**

Here, a_k= 1, for $0 \leq$ k $\leq \infty$.

Hence, G(x) = $1 + x + x^2 + x^3 + \ldots\ldots.. = 1(1 - x)$.

9. **Find the coefficient of $17x$ in the expansion of $(1 + x^5 + x^7)$20**

The only way to form an $17x$ term is to gather two $5x$ and one $7x$. Since there are 20C2 =190 ways to choose two $5x$ from the 20 multiplicands and 18 ways to choose one $7x$ from the remaining 18 multiplicands, the answer is 190* 18= 3420.

10. .**Explain all the three Methods to Solving Recurrences?**

Iteration: Start with the recurrence and keep applying the recurrence equation until we get a pattern. The result is a guess at the closed form.

Substitution: Guess the solution; prove it using induction. The result here is a proven closed form. It's often difficult to come up the guess so, in practice, iteration and substitution are used hand-in-hand.

Master Theorem: Plugging into a formula that gives an approximate bound on the solution. The result here is only a bound on the closed form. It is not an exact solution.

11.**If we have a recurrence relation for a sequence, is it possible to express the sequence in a way that does not use recursion?**

Sometimes. When we are able to do so, we find what is called the closed form of the recurrence. It is an algebraic formula or a definition that tells us how to find the nth term without needing to know any of the Preceding terms. The process of finding the closed form is called solving a recurrence.

12. **Find a recurrence relation for the sequence $8, \frac{24}{7}, \frac{72}{49}, \frac{216}{343}, \ldots$**

$$Here\ a_0 = 8\ and\ each\ term\ is\ multiplied\ by\ 3/7\ so\ an = (3/7)a_n - 1\ for\ n > 1.$$

13. A bank pays 6% (annual) intrest on savings, compounding the interest monthly. If Raj deposits Rs.1000/- on the first day of May, how much will he deposit be a worth year later?

Interest is 6% annually, so monthly is 6%/12=0.5% ==0.005

$P_{n+1}=P_n+0.005P_n$.

14. Define Catalan number.

In combinatorial mathematics, the Catalan numbers form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects.

14.For the alphabet = {0,1,2,3}, how many strings of length n contains an even number of 1's

SOLUTION:

Consider the nth symbol of a string of length n

(1) The nth symbol is 0, 2, 3. 3an-1

(2) The nth symbol is 1. Then there must be an odd number of 1's among the first n-1 symbols.

4n-1-an-1

an=3an-1+ (4n-1-an-1) =2 an-1+4n-1

15.Solve the recurrence relation an+2-4an-1+3an=-200 for n≥0 and a0=3000 and a1=3300

The solution for an+2-4an-1+3an=0 is an (h)=c1(3n)+c2.

(Let an (p) =An) The particular solution for an+2-4an-1+3an=-200 is an(p)=100n.

The solution to the problem is an= c1 (3n) +c2+100n

Finally, we have an= 100(3n) +2900+100n

16. Solve an = an-1 +8an-2 - 12an-3, with initial conditions a0 = 2, a1 = 3, and a2 = 19.

$$g(x) - xg(x) - 8x^2g(x) + 12x^3g(x) = 2 + x.(i.e), g(x) = (2 + x)/(1 - x - 8x^2 + 12x^3.$$

17 .State the properties of generating functions.

SOLUTION:

(i) For all $n \geq 0, P_d(n) = P_0(n)$. (i.e.,) The number of partitions of n into odd summands equals number of partitions of n into distinct summands.

(ii) The number of partitions of a positive integer n, where no summands appears more than twice equals number of partitions of a n where no summands is divisible by 3.

18. Show that the Fibonacci number fn is even if and only if n is a multiple of 3.

SOLUTION:

f1 = f2 = 1 is odd and f3 = 2 is even. Assume that f3k is even,
f3k−2 and f3k−1 are odd. Then f3k+1 = f3k + f3k−1 is odd (even + odd = odd),
and subsequently, f3k+2 = f3k+1 +f3k is also odd (odd+even = odd). It follows
that f3(k+1) = f3k+2 + f3k+1 is even (odd + odd = even).

19. **Find the sequence (xn) satisfying the recurrence relation**
xn = 2xn−1 + xn−2 − 2xn−3, n ≥ 3
and the initial conditions x0 = 1, x1 = 2, and x2 = 0.

SOLUTION.:

The characteristic equation of the recurrence relation is

20. **Define Exponential generating functions.**

An exponential generating function for the integer sequence, It is a function such that

PART-B

1. **Find the solution to the recurrence relation**
   $\boldsymbol{a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}}$ **with initial conditions**
   $\boldsymbol{a_0 = 1, a_1 = -2 \text{ and } a_2 = -1.}$

   Solution:

Given $a_n + 3a_{n-1} + 3a_{n-2} + a_{n-3} = 0$…………(1)

Let $a_n = r^n$ be a solution of (1)

$$r^n + 3r^{n-1} + 3r^{n-2} + r^{n-3} = 0$$

$$\Rightarrow r^n \left[1 + \frac{3}{r} + \frac{3}{r^2} + \frac{1}{r^3}\right] = 0$$

$$\Rightarrow r^n \frac{[r^3 + 3r^2 + 3r + 1]}{r^3} = 0$$

∴ The characteristic equation is $r^3 + 3r^2 + 3r + 1 = 0$

(i.e.,) $(r+1)^3 = 0$

$\Rightarrow r = -1, -1, -1.$

Hence $a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2(-1)^n$…………(2)

Given: $a_0 = 1 \Rightarrow a_0 = \alpha_1(-1)^0 + \alpha_2(0)(-1)^0 + \alpha_3 0^2(-1)^0 = 1$

$\Rightarrow \alpha_1 = 1$

Given: $a_1 = -2 \Rightarrow a_1 = \alpha_1(-1)^1 + \alpha_2(1)(-1)^1 + \alpha_3 1^2(-1)^1 = -2$

$\Rightarrow -\alpha_1 - \alpha_2 - \alpha_3 = -2$

$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 2$

$\Rightarrow \alpha_2 + \alpha_3 = 2 - 1$

$\Rightarrow \alpha_2 + \alpha_3 = 1$

Given: $a_2 = -1 \Rightarrow a_2 = \alpha_1(-1)^1 + \alpha_2(-1)(-1)^{-1} + \alpha_3(-1)^2(-1)^{-1} = -1$

$\Rightarrow -\alpha_1 + \alpha_2 - \alpha_3 = -1$

$\Rightarrow -1 + \alpha_2 - \alpha_3 = -1$

$\Rightarrow \alpha_2 - \alpha_3 = 0$………………(3)

$(2) + (3) \Rightarrow 2\alpha_2 = 1 \Rightarrow \alpha_2 = \frac{1}{2}$

$(3) \Rightarrow \alpha_3 = \alpha_2 \Rightarrow \alpha_3 = \frac{1}{2}$

$\therefore$ $(2) \Rightarrow a_n = (-1)^n + \frac{1}{2}n(-1)^n + \frac{1}{2}n^2(-1)^n.$

2. **Use generating functions to solve the recurrence relation** $\boldsymbol{a_n = 3a_{n-1} + 2}$ **with the initial condition** $\boldsymbol{a_0 = 1}$**.**

   Solution:

   Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$……………..(1)

   Where G(x) is the Generating Function for the sequence $\{a_n\}$.

   Given: $a_n = 3a_{n-1} + 2$

   $\Rightarrow \sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1}x^n + 2\sum_{n=1}^{\infty} x^n$

   $\Rightarrow G(x) - a_0 = 3xG(x) + 2\left[\frac{1}{1-x} - 1\right]$ $by\ (1)$

   Since $xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n$

   $\Rightarrow G(x) - 3xG(x) = 1 + 2\left[\frac{1}{1-x} - 1\right]$

   $\Rightarrow G(x)[1 - 3x] = 1 + 2\left[\frac{1-1+x}{1-x}\right]$

   $= 1 + \frac{2x}{1-x}$

   $= \frac{1-x+2x}{1-x}$

   $= \frac{1+x}{1-x}$

   $\Rightarrow G(x) = \frac{1+x}{(1-x)(1-3x)}$……………(2)

   Let $G(x) = \frac{1+x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x} = \frac{2}{(1-3x)} - \frac{1}{(1-x)}$

   $\Rightarrow \sum_{n=0}^{\infty} a_n x^n = 2\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} x^n$

   Hence $a_n = (2)3^n - 1$ which is the required solution.

3. **Find an explicit formula for the Fibonacci numbers.**

   Solution:

   The sequence of Fibonacci numbers satisfies the recurrence relation

   $f_n = f_{n-1} + f_{n-2}$ ……………………(1)

   And satisfies the initial conditions $f_0 = 0$ & $f_1 = 1$

   $(1) \Rightarrow f_n - f_{n-1} - f_{n-2} = 0$………………..(2)

   Let $f_n = r^n$ be a solution of the given equation

   $(2) \Rightarrow r^n - r^{n-1} - r^{n-2} = 0$

   $\Rightarrow r^n\left[1 - \frac{1}{r} - \frac{1}{r^2}\right] = 0$

   $\therefore$ The characteristic equation is $r^2 - r - 1 = 0$

   $r = \frac{1 \mp \sqrt{(-1)^2 - 4(-1)}}{2}$

$$r = \frac{1 \mp \sqrt{1+4}}{2}$$

$$\Rightarrow r_1 = \frac{1+\sqrt{5}}{2},\ r_2 = \frac{1-\sqrt{5}}{2}$$

$\therefore$ By Theorem,

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$f_0 = 0 \Rightarrow f_0 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$$

$$\Rightarrow \alpha_1 + \alpha_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$f_1 = 1 \Rightarrow f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

$$(1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 2 \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$(4)X(1+\sqrt{5}) \Rightarrow (1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 0 \ldots\ldots\ldots(6)$$

$$(5)X1 \qquad \Rightarrow (1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2 = 2 \ldots\ldots\ldots(7)$$

$$(6)-(7) \qquad \Rightarrow 2\sqrt{5}\alpha_2 = -2$$

$$\alpha_2 = \frac{-1}{\sqrt{5}}$$

$$(4) \qquad \Rightarrow \qquad \alpha_1 = \frac{1}{\sqrt{5}}$$

$$(3) \qquad \Rightarrow \qquad f_n = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

4. Explain generating Functions.

SOLUTION:

By definition, the *(ordinary) generating function* of the sequence $\{a_n\}$, where, by convention, the index n ranges from 0 to ∞, is a formal series

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$1/(1 - x) = 1 + x + x^2 + x^3 + \ldots$

5. Explain the procedure for solving recurrence relation.

SOLUTION:

A procedure for solving recursions.

Here is a six step procedure for solving recursions.

It is not guaranteed to work because it may not be possible to carry out all of the steps. Let the sequence be an.

1. Adjust the recursion so that it is valid for all n. In particular, an should be defined for all n

and an = 0 for n < 0. You may need to introduce a "correcting" sequence C_n

2. Introduce the generating function A(x) = P, n≥0 ,and n.

3. Substitute the recursion into the summation for A(x).

4. Rearrange the result so that you can recognize other occurrences of A(x) and so get rid of

summations.

5. If possible, solve the resulting equation to obtain an explicit formula for A(x).

6. By partial fractions, Taylor's Theorem or whatever, obtain an expression for the coefficient of

$x_n$in this explicit formula for A(x). This is a_n.

5.In a version of the computer language BASIC, the name of a variable is a string of one or two alpha numeric characters, where upper case and lower case letters are not distinguished.(An alpha numeric character is either one of the 26 English letters or one of the 10 digits).Moreover, a variable name must begin with a letter and must be different from the five string of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC ?

SOLUTION:

Let V equal the number of different variable names in this version of BASIC.
Let V1 be the number of these that are one character long and V2 be the number of these that are two character long.
Then by the sum rule, V = V1 + V2.
Note that V1 = 26, since a one-character variable name must be a letter.
Furthermore, by the product rule there are 26 . 36 strings of length two that begin with a letter and end with an alphanumeric character.
However, five of these are excluded, so that
V2 = 26 . 36 – 5 = 931.
Hence, there are V = V1 + V2 = 26 + 931 = 957 different names for variables in this version of BASIC.

6 **In how many ways can we draw a heart or a spade from an ordinary deck of playing cards ? A heart or an ace ? An ace or a king ? A card numbered 2 through 10 ? A numbered card or a king ?**

SOLUTION:

Since there are 13 hearts and 13 spades we may draw a heart or a spade in 13 + 13 = 26 ways.
We may draw a heart or an ace in 13 + 3 = 16 ways, since there are only 3 aces that are not hearts.
We may draw an ace or a king in 4 + 4 = 8 ways.
These are 9 cards numbered 2 through 10 in each of 4 suits, clubs, diamonds, hearts, or spades.
So we may choose a numbered card in 36 ways.
Thus, we may choose a numbered card or a king in 36 + 4 = 40 ways.

8. **Find the sum of all the four-digit numbers that can be formed with the digits 3, 2, 3 and 4.**

SOLUTION:

One thing is worth noticing here that a four-digit number so formed does not conatin a repeated digit except the digit 3.
This is implied from the question, because if it were not so, 3 should not have been repeated in the list of the digits.
To find the sum of the four-digit numbers formed with 3, 2, 3 and 4, we have to calculate the sum of digits at unit place in all such numbers.
The sum of the digits at ten, hundred and thousand place will be the same, only theirs place value will change.
The number of four-digit numbers in which 2 appears at unit place is determined by the number of permutations of digits 3, 3 and 4 to fill ten, hundred and thousand place. And this is3!/2!=3.
Similarly, the number of four-digit numbers in which 3 appears at unit place is 3 ! = 6.
The number of four-digit numbers in which 4 appears at unit place is 3!/2!=3.
Therefore, sum of the digits in the unit place of all the numbers = 3 × 2 + 6 × 3 + 3 × 4 = 36.
As stated above, the sum of the digits in all such numbers at ten, hundred and thousand places is 36 each.
Thus, the sum of all such numbers
= 36 × 1000 + 36 × 100 + 36 × 10 + 36
= 39996.

9. How many integer solutions are there for the equation $c1 + c2 + c3 + c4 = 25$ if $0 \leq c_i$ for all $1 \leq i \leq 4$?

SOLUTION:

For each child the possibilities can be described by the polynomial

$$1 + x + x2 + x3 + \ldots\ldots + x25$$

Then the answer to this problem is the coefficient of x25 in the generating function

$$f(x) = (1 + x + x2 + \ldots\ldots + x25)^4$$

The answer can also be obtained as the coefficient of x25 in the generating function

$$\underline{g(x) = (1 + x + x2 + x3 + \ldots\ldots + x25 + x26 + \ldots\ldots)^4}$$

10. Explain partition with an example.

SOLUTION:

In number theory and combinatorics, a **partition** of a positive integer *n*, also called an **integer partition**, is a way of writing *n* as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. (If order matters, the sum becomes a composition.) For example, 4 can be partitioned in five distinct ways:

4
3 + 1
2 + 2
2 + 1 + 1
1 + 1 + 1 + 1

The order-dependent composition 1 + 3 is the same partition as 3 + 1, while the two distinct compositions 1 + 2 + 1 and 1 + 1 + 2 represent the same partition 2 + 1 + 1.

A summand in a partition is also called a **part**. The number of partitions of *n* is given by the partition function*p*(*n*). So *p*(4) = 5. The notation $\lambda \vdash n$ means that λ is a partition of *n*.

Partitions can be graphically visualized with Young diagrams or Ferrers diagrams. They occur in a number of branches of mathematics and physics, including the study of symmetric polynomials, the symmetric group and in group representation theory in general.

Example:

The seven partitions of 5 are:

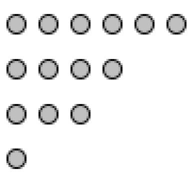
- 5
- 4 + 1
- 3 + 2
- 3 + 1 + 1
- 2 + 2 + 1
- 2 + 1 + 1 + 1
- 1 + 1 + 1 + 1 + 1

REPRESENTATION OF PARTITION

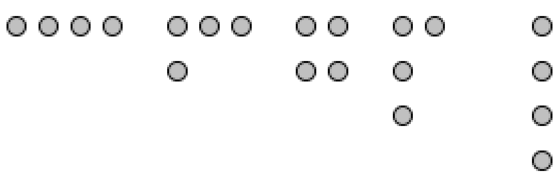
There are two common diagrammatic methods to represent partitions: as Ferrers diagrams, named after Norman Macleod Ferrers, and as Young diagrams, named after the British mathematician Alfred Young. Both have several possible conventions; here, we use *English notation*, with diagrams aligned in the upper-left corner.

Ferrers diagram

The partition 6 + 4 + 3 + 1 of the positive number 14 can be represented by the following diagram:



The 14 circles are lined up in 4 rows, each having the size of a part of the partition. The diagrams for the 5 partitions of the number 4 are listed below:



4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1